LAGRANGIAN QUANTUM HOMOLOGY

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Dedicated to Yasha Eliashberg on the occasion of his 60'th birthday

Contents

1. Introduction

The present paper is mainly a survey of our work [12] and [13] but it also contains the announcement of some new results. Its main purpose is to present an accessible introduction to a technique allowing efficient calculations in Lagrangian Floer theory.

This technique is based on counting elements in 0-dimensional moduli spaces formed by configurations consisting of pseudo-holomorphic disks joined together by Morse trajectories. In some form, such configurations have first appeared in the work of Oh in [33] and have been used in a more general setting in [18]. There are two basic reasons why such configurations are natural in this context.

First, if one tries to develop quantum homology and additional operations in the Lagrangian setting one needs to introduce a mechanism which compensates for the bubbling of disks as this is a co-dimension one phenomenon. The second reason is that the lens

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through which the topology of manifolds is understood algebraically is algebraic topology and this, via classical Morse theory, can be seen as the combinatorics of Morse trajectories. It is thus completely natural to approach symplectic topology and the topology of Lagrangians via the combinatorics of, so-called, pearly trajectories - schematically, these are just Morse trajectories with a finite number of points replaced by J-holomorphic curves.

As will be discussed below, in the case of monotone Lagrangians with minimal Maslov number at least 2, this idea can be fully implemented while dealing with the technical transversality issues in a relatively elementary way. The end result is a machinery which is effective in computations and which leads to several applications.

The paper is structured as follows. The second section reviews the construction of the quantum homology $QH(L)$ of a monotone Lagrangian $L \subset (M^{2n}, \omega)$ as an algebra over the quantum homology $QH(M)$ of the ambient manifold. The main ideas necessary to prove the properties of $QH(L)$ are described in §3. In §4 some additional useful structures are presented. We emphasize that in our (monotone) setting, as is well-known since the work of Oh [30], the Floer homology $HF(L, L)$ is well defined. Moreover, with appropriate coefficients, $QH(L)$ is isomorphic to $HF(L, L)$ and some of the structures that we define in "pearly" terms for $QH(L)$ are identified by this isomorphism to structures that are already known for $HF(L, L)$. The key point however is that, in applications, the "pearly" description of these operations is, by far, the most efficient one. This will become apparent by going over the examples of applications which are presented in the last four sections of the paper.

Acknowledgments. Some of the results of this paper have been announced at the Yasha Fest 2007 at Stanford University. We would like to thank the organizers for the opportunity to present our work there. We thank Yasha Eliashberg for many years of inspiration, both mathematical as well as non-mathematical.

2. The algebraic structures

2.1. **Setting.** All our symplectic manifolds will be implicitly assumed to be connected and tame (see [4]). The main examples of such manifolds are closed symplectic manifolds, manifolds which are symplectically convex at infinity as well as products of such. We denote by $\mathcal{J}$ the space of ω-compatible almost complex structures on M for which $(M, g_{\omega,J})$ is geometrically bounded, where $g_{\omega,J}$ is the associated Riemannian metric.

Lagrangian submanifolds $L \subset (M, \omega)$ will be assumed to be connected and closed. We denote by $H_2^D(M, L) \subset H_2(M, L)$ the image of the Hurewicz homomorphisms $\pi_2(M, L) \longrightarrow$

$H_2(M, L)$. We will be interested in *monotone* Lagrangians. This means that the two homomorphisms:

$$\omega : H_2^D(M, L) \longrightarrow \mathbb{Z}, \quad \mu : H_2^D(M, L) \longrightarrow \mathbb{R}$$

given respectively by integration of ω, $A \mapsto \int_A \omega$, and by the Maslov index satisfy:

$$\omega(A) > 0 \quad \text{iff} \quad \mu(A) > 0, \quad \forall\, A \in H_2^D(M, L).$$

It is easy to see that this is equivalent to the existence of a constant $\tau > 0$ such that

$$\omega(A) = \tau\mu(A), \ \forall\, A \in H_2^D(M, L)\ . \tag{1}$$

We refer to τ as the *monotonicity constant* of $L \subset (M, \omega)$. Define the *minimal Maslov number* of L to be the integer

$$N_L = \min\{\mu(A) > 0 \mid A \in H_2^D(M, L)\}.$$

Throughout this paper we assume that L is monotone with $N_L \geq 2$. Since the Maslov numbers come in multiples of N_L we will use sometimes the following notation:

$$\bar{\mu} = \tfrac{1}{N_L}\mu : H_2^D(M, L) \longrightarrow \mathbb{Z}. \tag{2}$$

Let $L \subset (M, \omega)$ be a monotone Lagrangian submanifold. Let $\Lambda = \mathbb{Z}_2[t^{-1}, t]$ be the ring of Laurent polynomials in t. We grade this ring so that $\deg t = -N_L$. Denote by $HF(L, L)$ the Floer homology of L with itself, defined over Λ. This is essentially the same homology as introduced by Oh [30, 31] only that since we work over Λ our $HF_*(L, L)$ has a relative $\mathbb{Z}$-grading (not a $\mathbb{Z}/N_L$-grading as in [30]) and is N_L-periodic in the sense that $HF_i(L, L) = HF_{i+N_L}(L, L) \cdot t$, $\forall i \in \mathbb{Z}$. See [12] for more details.

2.2. **Conventions from Morse theory.** Let f be a Morse function on a manifold and ρ a Riemannian metric. In case the manifold is not compact we will implicitly assume f to be proper, bounded below and with finitely many critical points. Denote by $\mathrm{Crit}(f)$ the set of critical points of f. For $x \in \mathrm{Crit}(f)$ we write $|x|$ for the Morse index of x.

We write ∇f for the gradient vector field of f with respect to ρ when the metric ρ is clear from the context. We will mostly work with the *negative* gradient flow of f, namely the flow of $-\nabla f$. We denote this flow by Φ_t, $-\infty \leq t \leq \infty$ (or $-\infty \leq t \leq \infty$ when the manifold is closed). In particular, all Morse homological constructions will be carried out using the *negative* gradient flow of the Morse function. For $x \in \mathrm{Crit}(f)$ we denote by $W_x^u(f)$, $W_x^s(f)$ the unstable and stable submanifolds of the flow Φ_t.

2.3. **The pearl complex.** Let $L \subset (M, \omega)$ be a monotone Lagrangian. Fix a triple (f, ρ, J) where $f : L \longrightarrow \mathbb{R}$ is a Morse function, ρ is a Riemannian metric on L and $J \in \mathcal{J}$. Define a complex generated by the critical points of f:

$$\mathcal{C}(f, \rho, J) = \mathbb{Z}_2 \langle \mathrm{Crit}(f) \rangle \otimes \Lambda.$$

We grade $\mathcal{C}(f, \rho, J)$ using the Morse indices of f and the grading of Λ mentioned above. In order to define a differential we need to introduce some moduli spaces.

Given two points $x, y \in L$ and a class $0 \neq A \in H_2^D(M, L)$ consider the space of all sequences $(u_1, \ldots, u_l)$ of every possible length $l \geq 1$, where:

(1) $u_i : (D, \partial D) \longrightarrow (M, L)$ is a *non-constant* J-holomorphic disk. Here and in what follows D stands for the closed unit disk in $\mathbb{C}$.
(2) There exists $-\infty \leq t' < 0$ such that $\Phi_{t'}(u_1(-1)) = x$.
(3) For every $1 \leq i \leq l-1$ there exists $0 < t_i < \infty$ such that $\Phi_{t_i}(u_i(1)) = u_{i+1}(-1)$.
(4) There exists $0 < t'' \leq \infty$ such that $\Phi_{t''}(u_l(1)) = y$.
(5) $[u_1] + \cdots + [u_l] = A$.

We view two elements in this space $(u_1, \ldots, u_l)$ and $(u'_1, \ldots, u'_{l'})$ as equivalent if $l = l'$ and for every $1 \leq i \leq l$ there exists $\sigma_i \in \mathrm{Aut}(D)$ with $\sigma_i(-1) = -1$, $\sigma_i(1) = 1$ and such that $u'_i = u_i \circ \sigma_i$. The space obtained from moding out by this equivalence relation is denoted by $\mathcal{P}_{\mathrm{prl}}(x, y; A; f, \rho, J)$. Elements of this space will be called *pearly trajectories connecting x to y*. A typical pearly trajectory is depicted in the left part of Figure 1.

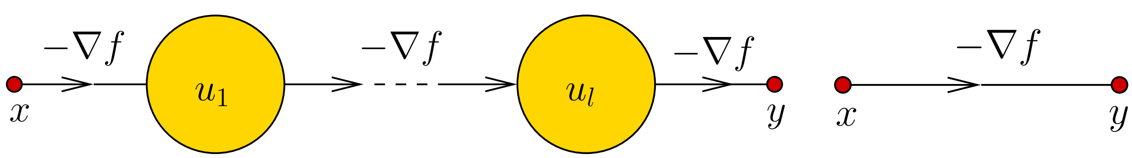

FIGURE 1. Pearly trajectories connecting x to y. On the left $A \neq 0$, on the right $A = 0$.

Most of the times we will be interested in the case when both x and y are critical points of f. Of course, in that case conditions (2), (4) above say that $u_1(-1) \in W_x^u(f)$ and $u_l(1) \in W_s^y(f)$ (in particular $t' = -\infty$, $t'' = \infty$). We extend the definition of the space of pearly trajectories to the case $A = 0$ by setting $\mathcal{P}_{\mathrm{prl}}(x, y; 0; f, \rho, J)$ to be the space of unparametrized trajectories of the negative gradient flow Φ_t connecting x to y. See the right part of Figure 1.

When $x, y \in \mathrm{Crit}(f)$ the virtual dimension of $\mathcal{P}_{\mathrm{prl}}(x, y; A; f, \rho, J)$ is:

$$\delta_{\mathrm{prl}}(x, y; A) = |x| - |y| + \mu(A) - 1. \tag{3}$$

Suppose that (f,ρ) is Morse-Smale and $\delta_{\mathrm{prl}}(x,y;A)=0$. It turns out that for a generic choice of J the space $\mathcal{P}_{\mathrm{prl}}(x,y;A;f,\rho,J)$ consists of a finite number of points (see §3.1 below). We denote by $\#_{\mathbb{Z}_2}\mathcal{P}(x,y;A;f,\rho,J)$ this number modulo 2. To define a differential $d:\mathcal{C}_*(f,\rho,J)\longrightarrow\mathcal{C}_{*-1}(f,\rho,J)$, fix a generic $J\in\mathcal{J}$. For $x\in\mathrm{Crit}(f)$ define:

$$d(x)=\sum_{y,A}\big(\#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}(x,y;A;f,\rho,J)\big)yt^{\bar{\mu}(A)}, \tag{4}$$

where the sum is taken over all pairs $y\in\mathrm{Crit}(f)$, $A\in H_2^D(M,L)$ with $\delta_{\mathrm{prl}}(x,y;A)=0$. Finally, extend d to $\mathcal{C}(f,\rho,J)$ by linearity over Λ.

Theorem 2.3.1. *The map d defined above is a differential, namely $d\circ d=0$. The homology of the complex $(\mathcal{C}_*(f,\rho,J),d)$, denoted $QH_*(L)$, is independent of the choice of the generic triple (f,ρ,J). More specifically, for every two generic triples $\mathscr{D}=(f,\rho,J)$, $\mathscr{D}'=(f',\rho',J')$ there exists a chain map $\psi_{\mathscr{D}',\mathscr{D}}:\mathcal{C}_*(\mathscr{D})\longrightarrow\mathcal{C}_*(\mathscr{D}')$ which descends to a canonical isomorphism in homology $\Psi_{\mathscr{D}',\mathscr{D}}:H_*(\mathcal{C}(\mathscr{D}))\longrightarrow H_*(\mathcal{C}(\mathscr{D}'))$. This systems of isomorphisms is compatible with composition: $\Psi_{\mathscr{D}'',\mathscr{D}'}\circ\Psi_{\mathscr{D}',\mathscr{D}}=\Psi_{\mathscr{D}'',\mathscr{D}}$, $\Psi_{\mathscr{D},\mathscr{D}}=\mathbb{1}$.*

Furthermore, there is an isomorphism $\Theta:HF_*(L,L)\longrightarrow QH_*(L)$ which is canonical up to a shift in grading.

We will refer to $QH(L)$ as the *quantum homology of L*. When we want to emphasize the specific choice of parameters (f,ρ,J) we will write $QH_*(L;f,\rho,J)$ for the homology of $\mathcal{C}_*(f,\rho,J)$. We call the canonical isomorphisms $\Psi_{\mathscr{D},\mathscr{D}'}$ *identification maps.*

2.4. **The Lagrangian quantum product.** Fix three Morse functions $f,f',f'':L\longrightarrow\mathbb{R}$, a Riemannian metric ρ on L and a generic $J\in\mathcal{J}$. We will now define an operation

$$\circ:\mathcal{C}(f,\rho,J)\otimes_\Lambda\mathcal{C}(f',\rho,J)\longrightarrow\mathcal{C}(f'',\rho,J),\quad x\otimes y\longmapsto x\circ y. \tag{5}$$

This operation will have degree $-n$, where $n=\dim L$, i.e. $\circ:\mathcal{C}_i(f,\rho,J)\otimes\mathcal{C}_j(f',\rho,J)\longrightarrow\mathcal{C}_{i+j-n}(f'',\rho,J)$ for every $i,j\in\mathbb{Z}$. For this end we have to introduce some other moduli spaces. Let $x\in\mathrm{Crit}(f)$, $y\in\mathrm{Crit}(f')$, $z\in\mathrm{Crit}(f'')$ and $A\in H_2^D(M,L)$. Consider the space of all tuples $(\mathbf{u},\mathbf{u}',\mathbf{u}'',v)$ where:

(1) $v:(D,\partial D)\longrightarrow(M,L)$ is a J-holomorphic disk (which is allowed to be constant).
(2) If we denote $\tilde{x}=v(e^{-2\pi i/3})$, $\tilde{y}=v(e^{2\pi i/3})$, $\tilde{z}=v(1)$ then:

$$\mathbf{u}\in\mathcal{P}_{\mathrm{prl}}(x,\tilde{x};B;J,\rho,f),\quad \mathbf{u}'\in\mathcal{P}_{\mathrm{prl}}(y,\tilde{y};B';J,\rho,f'),\quad \mathbf{u}''\in\mathcal{P}_{\mathrm{prl}}(\tilde{z},z;B'';J,\rho,f''),$$

for some $B,B',B''\in H_2^D(M,L)$.
(3) $B+B'+B''+[v]=A$.

Elements of this space will be denoted by $\mathcal{P}_{\mathrm{prod}}(x,y,z;A;f,f',f'',\rho,J)$. A typical element is depicted in figure 2.

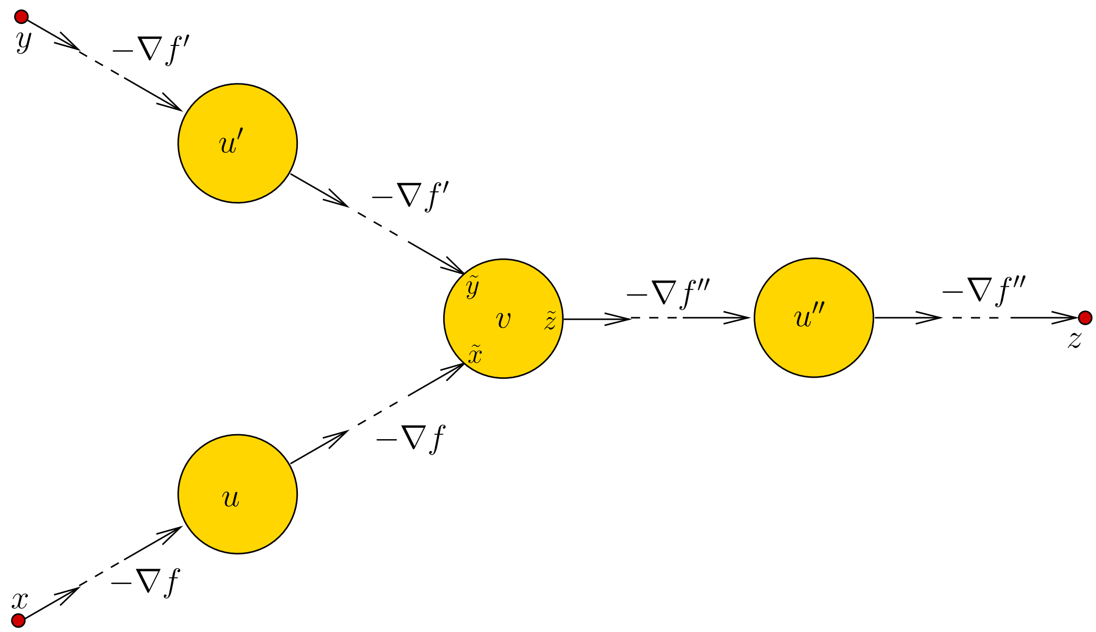

FIGURE 2. An element of $\mathcal{P}_{\text{prod}}(x, y, z; A; f, f', f'', \rho, J)$.

The virtual dimension of $\mathcal{P}_{\text{prod}}(x, y, z; A; f, f', f'', \rho, J)$ is

$$\delta_{\text{prod}}(x, y, z; A) = |x| + |y| - |z| - n + \mu(A). \tag{6}$$

Suppose that (f, f', f'', ρ) are in general position and $\delta_{\text{prod}}(x, y, z; A) = 0$. It turns out that for generic $J \in \mathcal{J}$ the space $\mathcal{P}_{\text{prod}}(x, y; A; f, \rho, J)$ consists of a finite number of points (see §3.1). The operation $\circ$ is now defined as follows. For $x \in \text{Crit}(f)$, $y \in \text{Crit}(f')$ put:

$$x \circ y = \sum_{z,A} \big(\#_{\mathbb{Z}_2} \mathcal{P}_{\text{prod}}(x, y, z; A; f, f', f'', \rho, J)\big) z t^{\bar{\mu}(A)}, \tag{7}$$

where the sum is taken over all $z \in \text{Crit}(f'')$ and $A \in H_2^D(M, L)$ with $\delta_{\text{prod}}(x, y, z; A) = 0$. Again, we extend the definition of $\circ$ by linearity over Λ.

Theorem 2.4.1. *i. The map $\circ$ is a chain map, hence descends to an operation in homology. The operation in homology is canonical in the sense that it is compatible with the system of identification maps $\Psi_{-,-}$ mentioned in Theorem 2.3.1. Thus we obtain a canonical operation, still denoted $\circ$:*

$$\circ : QH_i(L) \otimes QH_j(L) \to QH_{i+j-n}(L), \quad \forall\, i, j \in \mathbb{Z}.$$

ii. The operation $\circ$ endows $QH(L)$ with the structure of an associative ring with unity. This ring is, in general, not commutative (not even in the graded sense).

iii. The unity of $QH(L)$ has degree n. In fact, if $f : L \to \mathbb{R}$ is a Morse function with exactly one (local) maximum $x \in L$ then $x \in \mathcal{C}_n(f, \rho, J)$ is a cycle whose homology

class $[x] \in QH_n(L)$ *does not depend on* (f, ρ, J) *and which represents the unity. By abuse of notation, and by analogy to Morse theory, we denote the unity by* $[L] \in QH_n(L)$.

iv. *The product* $\circ$ *corresponds under the identification* $\Theta : HF_*(L, L) \to QH_*(L)$ *to the Donaldson product defined by counting holomorphic triangles.*

2.5. **The quantum module structure.** Here we define an external operation which makes $QH(L)$ a module over the quantum homology of the ambient manifold.

We start with a few preliminaries on quantum homology. First recall that if $L \subset (M, \omega)$ is monotone then the ambient symplectic manifold (M, ω) is spherically monotone, namely there exists a constant $\nu > 0$ such that $\omega(A) = \nu c_1(A)$ for every $A \in \pi_2(M)$, where $c_1 \in H^2(M)$ is the first Chern class of the tangent bundle of M. In fact the monotonicity constant ν is related to τ (see (1)) by $\nu = 2\tau$. We denote by C_M the minimal Chern number of M:

$$C_M = \min\{c_1(A) > 0 \mid A \in \pi_2(M)\}.$$

Let $\Gamma = \mathbb{Z}_2[s, s^{-1}]$. Define a grading on Γ by setting $\deg s = -2C_M$. A special convention is valid if $c_1|_{\pi_2(M)} = 0$. In this case, we put $C_M = \infty$ and $\Gamma = \mathbb{Z}_2$.

Denote by $QH(M) = H(M; \mathbb{Z}_2) \otimes \Gamma$ the quantum homology of M endowed with the quantum intersection product $* : QH_l(M) \otimes QH_k(M) \longrightarrow QH_{l+k-2n}(M)$, where $2n = \dim M$. Recall that this is an associative and commutative product (we work over $\mathbb{Z}_2$). The unity is the fundamental class $[M] \in QH_{2n}(M)$. We refer to [29] for the foundations of quantum homology theory.

We will actually need to work with the following small extension of $QH(M)$. Consider the ring embedding $\Gamma \hookrightarrow \Lambda$ induced by $s \mapsto t^{2C_M/N_L}$. Using this embedding we can regard Λ as a module over Γ. Define

$$QH(M; \Lambda) = QH(M) \otimes_\Gamma \Lambda.$$

We endow $QH(M; \Lambda)$ with the same quantum intersection product $*$.

Example 2.5.1. Consider $M = \mathbb{C}P^n$ endowed with its standard Kähler symplectic form. This manifold is monotone with $C_M = n + 1$. Denote by $h = [\mathbb{C}P^{n-1}] \in H_{2n-2}(\mathbb{C}P^n)$ the homology class of a hyperplane and by $h^{\cap j} = [\mathbb{C}P^{n-j}] \in H_{2n-2j}(\mathbb{C}P^n)$ the class of a codimension j complex linear subspace. The quantum product in $QH(\mathbb{C}P^n)$ is given by:

$$h^{*j} = \begin{cases} h^{\cap j}, & 0 \le j \le n \\ [\mathbb{C}P^n]s, & j = n+1 \end{cases} \tag{8}$$

On the other hand, if we work for example with Lagrangians L with $N_L = n+1$ (e.g. $L = \mathbb{R}P^n \subset \mathbb{C}P^n$) then in Λ we have $\deg t = -(n+1)$ and the embedding $\Gamma \hookrightarrow \Lambda$ is given by $s \mapsto t^2$. Thus the last identity in (8) becomes in $QH(\mathbb{C}P^n; \Lambda)$: $h^{*(n+1)} = [\mathbb{C}P^n]t^2$.

We proceed with the definition of the module action of $QH(M;\Lambda)$ on $QH(L)$. Let $f : L \longrightarrow \mathbb{R}$, $h : M \longrightarrow \mathbb{R}$ be Morse functions and ρ_L, ρ_M Riemannian metrics on L and M. We write Φ_t^f and Φ_t^h for the negative gradient flows of f and h with respect to the corresponding Riemannian metrics. Denote by $C(h, \rho_M; \Lambda) = \mathbb{Z}_2\langle \mathrm{Crit}(h)\rangle \otimes \Lambda$ the Morse complex with coefficients in Λ. Clearly there is an isomorphism of Λ-modules: $H_*(C(h,\rho;\Lambda)) \cong QH_*(M;\Lambda)$.

Let $x, y \in \mathrm{Crit}(f)$, $a \in \mathrm{Crit}(h)$, $A \in H_2^D(M,L)$ (we allow A to be 0 here). Consider the space of all sequences $(u_1, \ldots, u_l; k)$, of every possible length $l \geq 1$, where:

(1) $1 \leq k \leq l$.
(2) $u_i : (D, \partial D) \longrightarrow (M, L)$ is a J-holomorphic disk for every $1 \leq i \leq l$, which is assumed to be non-constant except possibly when $i = k$.
(3) $u_1(-1) \in W_x^u(f)$.
(4) For every $1 \leq i \leq l-1$ there exists $0 < t_i < \infty$ such that $\Phi_{t_i}^f(u_i(1)) = u_{i+1}(-1)$.
(5) $u_l(1) \in W_s^y(f)$.
(6) $u_k(0) \in W_a^u(h)$.
(7) $[u_1] + \cdots + [u_l] = A$.

We view two elements in this space $(u_1, \ldots, u_l; k)$ and $(u'_1, \ldots, u'_{l'}; k')$ as equivalent if $l = l'$, $k = k'$, and for every $i \neq k$ there exists $\sigma_i \in \mathrm{Aut}(D)$ with $\sigma_i(-1) = -1$, $\sigma_i(1) = 1$ and such that $u'_i = u_i \circ \sigma_i$. The space obtained by moding out by this equivalence relation is denoted by $\mathcal{P}_{\mathrm{mod}}(a, x, y; A; h, \rho_M, f, \rho_L, J)$. A typical element of this space is depicted in Figure 3.

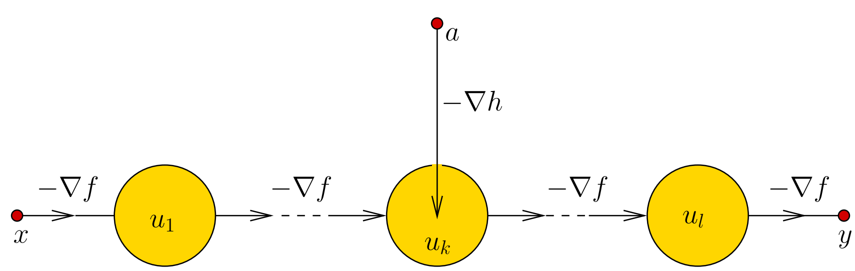

FIGURE 3. An element of $\mathcal{P}_{\mathrm{mod}}(a, x, y; A; h, \rho_M, f, \rho_L, J)$.

The virtual dimension of $\mathcal{P}_{\mathrm{mod}}(a, x, y; A; h, \rho_M, f, \rho_L, J)$ is:

$$\delta_{\mathrm{mod}}(a, x, y; A) = |a| + |x| - |y| + \mu(A) - 2n. \tag{9}$$

As before, if (h, ρ_M, f, ρ_L) are in general position and $J \in \mathcal{J}$ is generic then whenever $\delta_{\text{mod}}(a, x, y; A) = 0$ the space $\mathcal{P}_{\text{mod}}(a, x, y; A; h, \rho_M, f, \rho_L, J)$ consists of a finite number of points.

We now define a map $\circledast : C(h, \rho_M; \Lambda) \otimes_\Lambda \mathcal{C}(f, \rho_L, J) \longrightarrow \mathcal{C}(f, \rho_L, J)$, $a \otimes x \longmapsto a \circledast x$. For $a \in \text{Crit}(h)$, $x \in \text{Crit}(f)$ put:

$$a \circledast x = \sum_{y,A} \#_{\mathbb{Z}_2}\big(\mathcal{P}_{\text{mod}}(a, x, y; A; h, \rho_M, f, \rho_L, J)\big) y t^{\bar{\mu}(A)}, \tag{10}$$

where the sum is taken over all pairs $y \in \text{Crit}(f)$, $A \in H_2^D(M, L)$ with $\delta_{\text{mod}}(a, x, y; A) = 0$. Finally, extend $\circledast$ by linearity over Λ. Note that the operation $\circledast$ has degree $-2n$, i.e. $\circledast : C_k(h, \rho_M; \Lambda) \otimes_\Lambda \mathcal{C}_j(f, \rho_L, J) \longrightarrow \mathcal{C}_{k+j-2n}(f, \rho_L, J)$.

Theorem 2.5.2. *i. The map $\circledast$ is a chain map, hence descends to a an operation in homology. This operation in homology is compatible with the identification maps $\Psi_{-,-}$ mentioned in Theorem 2.3.1 as well as with the Morse homological identifications for the homology $QH(M; \Lambda)$. Thus we obtain a canonical operation, still denoted $\circledast$:*

$$\circledast : QH_k(M; \Lambda) \otimes QH_j(L) \longrightarrow QH_{k+j-2n}(L), \quad \forall\, k, j \in \mathbb{Z}.$$

ii. The operation $\circledast$ makes $QH(L)$ into module over the ring $QH(M; \Lambda)$ when the latter is endowed with its quantum product $*$. This means, in particular, that the following identities hold (in homology):

$$a \circledast (b \circledast x) = (a * b) \circledast x, \quad [M] \circledast x = x,$$

for every homology classes $a, b \in QH(M; \Lambda)$, $x \in QH(L)$.

iii. Furthermore, the ring $QH(L)$ endowed with the product $\circ$ (see Theorem 2.4.1), becomes a two-sided algebra over $QH(M)$. This means that we have the following additional identities (in homology):

$$a \circledast (x \circ y) = (a \circledast x) \circ y = x \circ (a \circledast y),$$

for every homology classes $a \in QH(M; \Lambda)$, $x, y \in QH(L)$.

Remark 2.5.3. The quantum homology ring $QH(L)$ is actually a symplectic invariant of L in the sense that if $\phi : M \to M$ is a symplectomorphism and $L' = \phi(L)$, then $QH(L) \cong QH(L')$. In case, $\phi \in \text{Symp}_H$, then this isomorphism is also an isomorphism of algebras (here, Symp_H is the group of symplectomorphisms of M which induce the identity in $H_*(M; \mathbb{Z}_2)$).

2.6. **The quantum inclusion map.** We now define a quantum version of the classical map $H_*(L) \longrightarrow H_*(M)$ induced by the inclusion.

As in §2.5 above, fix Morse functions $h : M \longrightarrow \mathbb{R}$, $f : L \longrightarrow \mathbb{R}$, Riemannian metrics ρ_M, ρ_L on M and L and an almost complex structure $J \in \mathcal{J}$. We use the same notation Φ^h_t, Φ^f_t for the negative gradient flows, as in §2.5.

For $x \in \mathrm{Crit}(f)$, $a \in \mathrm{Crit}(h)$ and $A \in H_2^D(M, L)$ consider the space of all sequences $(u_1, \ldots, u_l)$ of every possible length $l \geq 1$ such that:

(1) $u_i : (D, \partial D) \longrightarrow (M, L)$ is a J-holomorphic disk for every $1 \leq i \leq l$. All the disks u_i, $1 \leq i \leq l-1$ are assumed to be non-constant, but u_l is allowed to be constant.
(2) $u_1(-1) \in W^u_x(f)$.
(3) For every $1 \leq i \leq l-1$ there exists $0 < t_i < \infty$ such that $\Phi^f_{t_i}(u_i(1)) = u_{i+1}(-1)$.
(4) $u_l(0) \in W^s_a(h)$.
(5) $[u_1] + \cdots + [u_l] = A$.

As before, we view two elements in this space $(u_1, \ldots, u_l)$ and $(u'_1, \ldots, u'_{l'})$ as equivalent if $l = l'$ and for every $1 \leq i \leq l-1$ there exists $\sigma_i \in \mathrm{Aut}(D)$ with $\sigma_i(-1) = -1$, $\sigma_i(1) = 1$ and such that $u'_i = u_i \circ \sigma_i$. The space obtained by moding out by this equivalence relation is denoted by $\mathcal{P}_{\mathrm{inc}}(x, a; A; h, \rho_M, f, \rho_L, J)$. A typical element of this space is depicted in Figure 4.

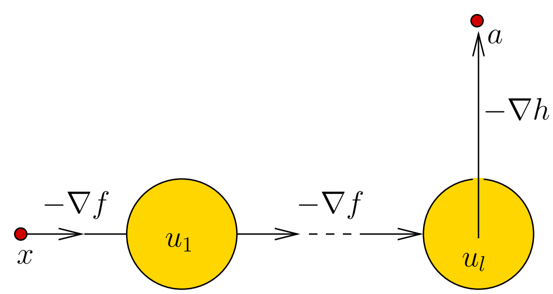

FIGURE 4. An element of $\mathcal{P}_{\mathrm{inc}}(x, a; A; h, \rho_M, f, \rho_L, J)$.

The virtual dimension of this space is:

$$\delta_{\mathrm{inc}}(x, a; A) = |x| - |a| + \mu(A). \tag{11}$$

As before, if (h, ρ_M, f, ρ_L) are in general position and $J \in \mathcal{J}$ is generic then whenever $\delta_{\mathrm{inc}}(x, a; A) = 0$ the space $\mathcal{P}_{\mathrm{mod}}(x, a; A; h, \rho_M, f, \rho_L, J)$ consists of a finite number of points.

We now define a map $\widetilde{i}_L : \mathcal{C}_*(f, \rho_L, J) \longrightarrow C_*(h, \rho_M; \Lambda)$ of degree 0 using the formula:

$$\widetilde{i}_L(x) = \sum_{a,A} \big(\#_{\mathbb{Z}_2} \mathcal{P}_{\mathrm{inc}}(x, a; A; h, \rho_M, f, \rho_L, J)\big) a t^{\bar{\mu}(A)}, \quad \forall\, x \in \mathrm{Crit}(f), \tag{12}$$

where the sum is taken over all pairs a, A with $\delta_{\text{inc}}(x, a; A) = 0$. We extend $\widetilde{i}_L$ to $\mathcal{C}(f, \rho_L, J)$ by linearity over Λ.

Theorem 2.6.1. *The map $\widetilde{i}_L$ is a chain map, hence descends to homology. The induced map in homology is compatible with the identifications maps $\Psi_{-,-}$ mentioned in Theorem 2.3.1 as well as with the Morse homological identifications for the homology $QH(M; \Lambda)$. Thus we obtain a canonical map*

$$i_L : QH_*(L) \longrightarrow QH_*(M; \Lambda).$$

Moreover, when viewing $QH(L)$ as a module over $QH(M; \Lambda)$ (see Theorem 2.5.2), i_L is a map of $QH_*(M; \Lambda)$-modules. In other other words, for every $a \in QH(M; \Lambda)$, $x \in QH(L)$ we have $i_L(a \circledast x) = a * i_L(x)$.

2.7. **Relation to the classical operations.** All the operations described in §2.3 - 2.6 have classical Morse-theoretic counterparts. For example, the pearly differential d can be written as a sum of operators $d = \partial_0 + \partial_1 t + \cdots + \partial_\nu t^\nu$, where $\partial_i : C_*(f, \rho) \to C_{*-1+iN_L}(f, \rho)$ is defined as:

$$\partial_i(x) = \sum_{\substack{y, A, \\ \mu(A) = iN_L, \\ |y| = |x| - 1 + iN_L}} \#_{\mathbb{Z}_2} \mathcal{P}_{\text{prl}}(x, y; A; f, \rho, J)\big) y.$$

While the operators ∂_i, $i \geq 1$ are in general not differentials the operator $\partial_0 : \mathcal{C}_* \to \mathcal{C}_{*-1}$ is precisely the Morse homology differential. To see this note that the only space $\mathcal{P}_{\text{prl}}(x, y; A; f, \rho, J)$ that contributes to ∂_0 is when $A = 0$. This follows from monotonicity since there are no pseudo-holomorphic disks with Maslov index 0 that are not constant. Thus $\partial_0(x)$ involves only the spaces $\mathcal{P}_{\text{prl}}(x, y; 0; f, \rho, J)$ which, by definition, are the spaces of negative gradient trajectories of f connecting x to y.

Similarly, the operation $\circ : \mathcal{C}(f, \rho, J) \otimes_\Lambda \mathcal{C}(f', \rho, J) \longrightarrow \mathcal{C}(f'', \rho, J)$ defined in §2.4 is related to the classical intersection product in Morse homology in the following way. Write $\circ$ as a sum:

$$x \circ y = x \circ_0 y + x \circ_1 yt + \cdots + x \circ_\kappa yt^\kappa,$$

where $\circ_i : C_p(f, \rho) \otimes C_q(f', \rho) \to C_{p+q-n-iN_L}(f'', \rho)$ stands for the coefficient in front of t^i in formula (7). The operator $\circ_0 : \mathcal{C}_p \otimes \mathcal{C}_q \to \mathcal{C}_{p+q-n}$ coincides with the Morse-theoretic intersection product. Indeed, by monotonicity $\circ_0$ involves only the spaces $\mathcal{P}_{\text{prod}}(x, y, z; 0; f, f', f'', \rho, J)$. Moreover, in this case every element $(\mathbf{u}, \mathbf{u}', \mathbf{u}'', v)$ must have $v = \text{const}$ and all the other pearly trajectories $\mathbf{u}, \mathbf{u}', \mathbf{u}''$ contain no disks. Thus, the points of $\mathcal{P}_{\text{prod}}(x, y, z; 0; f, f', f'', \rho, J)$ are in 1–1 correspondence with points of the triple intersection $W^u_x(f) \cap W^u_y(f') \cap W^s_z(f'')$. This is precisely the Morse-theoretic definition of the intersection product on the chain level.

The quantum module structure of §2.5 is related to the external intersection product, intersecting cycles in M with cycles in L. Indeed, if we take $A = 0$ in the definition of $\mathcal{P}_{\text{mod}}(a, x, y; A; h, \rho_M, f, \rho_L, J)$ we see that every element $(u_1, \ldots, u_l; k)$ in this space must have $l = 1$ and the disk u_1 must be constant. These elements are in 1–1 correspondence with the points of the triple intersection $W_x^u(f) \cap W_y^s(f) \cap W_a^u(h)$.

Finally, the quantum inclusion from §2.6 is related in a similar way to the classical inclusion map sending cycles in L to cycles in M.

The relation to the classical operation bears some analogy to the situation in the theory of quantum homology (of the ambient symplectic manifold). However a bit of caution is necessary here: this analogy holds on the chain level but not in homology. In fact, there is no way to recover the singular homology $H_*(L)$ from the quantum homology $QH_*(L)$. Similarly, while the (ambient) quantum product on $QH(M)$ can be seen as a deformation of the classical intersection product this is not the case for $QH(L)$. For example, there are situations in which $QH(L)$ vanishes (e.g. when L is displaceable). The reason is that the pearly differential d is already deformed with respect to the Morse differential ∂_0 hence the relation between $QH_*(L)$ and $H_*(L)$ is more complicated. In fact, $QH_*(L)$ and $H_*(L)$ are related via a spectral sequence whose second page can be constructed from $H_*(L)$. This spectral sequence was introduced by Oh [32]. See also [10, 14] for an alternative description and applications of this point of view. In §4.1.2 we will briefly review this construction.

In §4.3 we will discuss further the relation between the quantum operations and the classical ones on the homological level.

2.8. **Previous works and related references.** Parts of the constructions above appear already in the literature and have been verified up to various degrees of rigor. The complex $\mathcal{C}(f, \rho, J)$ was first introduced by Oh [33] (see also Fukaya [21]) and is a particular case of the cluster complex as described in Cornea-Lalonde [18]. The module structure is probably known to experts – at least in the Floer homology setting – but has not been explicitly described yet in the literature. The quantum product which is a variant of the Donaldson product might not be widely known in the form presented above. The quantum inclusion map i_L is the analogue of a map first studied by Albers in [3] in the absence of bubbling. The comparison map Θ from Theorem 2.3.1 is an extension of the Piunikin-Salamon-Schwarz construction [34], it extends also the partial map constructed by Albers in [2] and a more general such map was described in [18] in the "cluster" context. We also remark that this comparison map identifies all the algebraic structures described above with the corresponding ones defined in terms of the Floer complex.

3. Main ideas for the proofs of the Theorems from §2

Most of the proofs of Theorems 2.3.1- 2.6.1 follow standard arguments from Morse and Floer theories, the main building blocks being: *transversality, compactness and gluing*. The scheme is roughly as follows. One considers the same moduli spaces introduced above but with virtual dimension 1. A transversality argument shows that for a generic choice of parameters these spaces are smooth 1-dimensional manifolds. These manifolds are in general not compact. Compactness and gluing are then used to give a precise description of the compactification of these 1-dimensional manifolds. It then turns out that these compactifications still have a structure of 1-dimensional manifolds with boundary. The boundary points can usually be described in terms of elements of the same types of moduli spaces, but now having virtual dimension 0. As the number of boundary points of a compact 1-dimensional manifold must be 0 mod 2 we obtain form this procedure an identity involving the number of points in various 0-dimensional moduli spaces. These identities, it turns out, are equivalent to the statements saying that d is a differential, and that the quantum operations $\circ$, $\circledast$, i_L are chain maps. The other properties stated in the Theorems above can be proved by a similar scheme by introducing appropriate moduli spaces, 0-dimensional as well as 1-dimensional.

Below we will outline in some detail the proof of the simplest statement: the fact that the map d is a differential, as stated in Theorem 2.3.1. Still, we will skip many technical points, and only mention the main ideas in each step. We refer the reader to [12, 13] for the precise details.

While compactness and gluing are rather standard by now, our approach to transversality is somewhat less mainstream. It will be explained in the next subsection. Throughout the rest of this section we continue to assume implicitly that $L \subset (M, \omega)$ is monotone.

3.1. **Transversality for pearly moduli spaces.** Formally we need (at least) four types of transversality results: one for each of the spaces $\mathcal{P}_{\text{prl}}$, $\mathcal{P}_{\text{prod}}$, $\mathcal{P}_{\text{mod}}$, $\mathcal{P}_{\text{inc}}$. The statements in all four cases are quite similar. They all assert that when the Morse functions, metric and almost complex structures are chosen generically then whenever the virtual dimension $\delta(\cdots)$ is ≤ 1, the corresponding moduli space $\mathcal{P}(\cdots)$ is a smooth manifold whose dimension equals the virtual dimension. Moreover, when $\delta(\cdots) = 0$ the corresponding space is a compact 0-dimensional manifold hence consists of a finite number of points.

In order not to make lengthy repetitions of similar statements we will use the following unifying notation. We will denote by $\mathcal{S}$ the type of the moduli space under considerations, namely $\mathcal{S}$ can be one of "prl", "prod", "mod" or "inc". We denote by $\mathcal{F}$ the choice of the Morse data and by I a tuple consisting of critical points and homology class $A \in H_2^D(M, L)$. More specifically:

(1) When $\mathcal{S} = \mathrm{prl}$, $\mathcal{F} = (f, \rho)$, $I = (x, y; A)$, where f is a Morse function on L, ρ is a Riemannian metric on L and $x, y \in \mathrm{Crit}(f)$.
(2) When $\mathcal{S} = \mathrm{prod}$, $\mathcal{F} = (f, f', f'', \rho)$, $I = (x, y, z; A)$, where f, f', f'' are Morse functions on L, ρ is a Riemannian metric on L and $x \in \mathrm{Crit}(f)$, $y \in \mathrm{Crit}(f')$, $z \in \mathrm{Crit}(f'')$.
(3) When $\mathcal{S} = \mathrm{mod}$, $\mathcal{F} = (h, \rho_M, f, \rho_L)$, $I = (a, x, y; A)$, where h, ρ_M, resp. f, ρ_L, are a Morse function and a Riemannian metric on M, resp. L, and $a \in \mathrm{Crit}(h)$, $x, y \in \mathrm{Crit}(f)$.
(4) When $\mathcal{S} = \mathrm{inc}$, $\mathcal{F} = (h, \rho_M, f, \rho_L)$, $I = (x, a; A)$, where the components of $\mathcal{F}$ as well as x, a are as in point 3 above.

We denote by $\delta_{\mathcal{S}}(I)$ the virtual dimension of the space $\mathcal{P}_{\mathcal{S}}(I, \mathcal{F}, J)$ as defined by formulae (3), (6), (9), (11) in §2.

We will have to impose some genericity assumptions on the Morse data $\mathcal{F}$. We will call $\mathcal{F}$ generic if the following holds:

Assumption 3.1.1 (Genericity)**.** When $\mathcal{S} = \mathrm{prl}$ assume that $\mathcal{F} = (f, \rho)$ is Morse-Smale. When $\mathcal{S} = \mathrm{prod}$ assume that $\mathcal{F} = (f, f', f'', \rho)$ has the property that for every critical point $p \in \mathrm{Crit}(f)$, $p' \in \mathrm{Crit}(f')$, $p'' \in \mathrm{Crit}(f'')$ the triple intersection $W^u_p(f) \cap W^u_{p'}(f') \cap W^s_{p''}(f'')$ is transverse. Finally, when $\mathcal{S} = \mathrm{mod}$ or inc assume that the following holds: each of the pairs (f, ρ_L) and (h, ρ_M) is Morse-Smale and, if M is compact, h has a single maximum. Furthermore:

a. In case M is not compact we assume that h is proper, bounded below and has finitely many critical points.
b. None of the critical points of h lies on L.
c. For every $a \in \mathrm{Crit}(h)$ the unstable submanifold $W^u_a(h)$ as well as the stable submanifold $W^s_a(h)$ are both transverse to L.
d. For every $a \in \mathrm{Crit}(h)$, $x, y \in \mathrm{Crit}(f)$, $W^u_a(h)$ is transverse to $W^u_x(f)$ and to $W^s_y(f)$.

Standard Morse theory arguments show that if $\mathcal{F}$ is generic in the usual sense, then it satisfies Assumption 3.1.1. Here is the transversality result needed to construct the structures in §2.3- 2.6 and to show that they induce the respective operations in homology.

Proposition 3.1.2. *Let $\mathcal{S}$ and $\mathcal{F}$ be as above. Assume that $\mathcal{F}$ satisfies the genericity assumption 3.1.1 and that, if $N_L = 2$, $\delta_{\mathcal{S}}(I) = 1$, then $\mathcal{S} \neq \mathrm{mod}$. Then there exists a second category subset $\mathcal{J}_{\mathrm{reg}} \subset \mathcal{J}$ such that for every $J \in \mathcal{J}_{\mathrm{reg}}$ the following holds. For every tuple I as above with $\delta_{\mathcal{S}}(I) \leq 1$ the space $\mathcal{P}_{\mathcal{S}}(I, \mathcal{F}, J)$ is either empty or a smooth manifold of dimension $\delta_{\mathcal{S}}(I)$. Moreover, when $\delta_{\mathcal{S}}(I) = 0$ this 0-dimensional manifold is compact, hence consists of a finite number of points.*

This transversality statement is emblematic for the types of arguments involved. However, it is not sufficient to also prove the relations - associativity etc - contained in the statements of 2.4 and 2.5 as well as to deal with the exceptional case $\mathcal{S} = \text{mod}, N_L = 2, \delta_{\mathcal{S}}(I) = 1$. New moduli spaces are needed for this purpose and Hamiltonian perturbations are required to show the fact that $QH(L)$ is an algebra over $QH(M;\Lambda)$ (see §3.5 for a more complete discussion of this).

3.1.1. *How to prove transversality.* In order to insure that moduli spaces involving pseudo-holomorphic curves are smooth manifolds, and that certain evaluation maps are transverse to some submanifolds, one has to restrict to curves $u : \Sigma \to M$ that are simple (or, at least, somewhere injective). Indeed, it is well known (see [29]) that for generic J the space of simple J-holomorphic curves (in a given class) is a smooth manifold whose dimension equals the virtual dimension. Moreover, for simple curves one can arrange all appropriate evaluation maps to be transverse to any given submanifold in their target.

Appearance of non-simple curves is relatively easy to deal with (at least in the monotone case) when the domains of the curves Σ are closed Riemann surfaces since a curve u that is not simple factors as $u' \circ \phi$ where $u' : \Sigma' \to M$ is a simple curve and $\phi : \Sigma \to \Sigma'$ is a branched covering (see [29]). One then replaces u by u' for which transversality holds.

The situation becomes more involved when the domain of the curves has boundary, as in our case, when Σ is a disk. It is well known that in this case a pseudo-holomorphic curve $u : (D, \partial D) \to (M, L)$ might not be simple yet not multiply covered in the sense of the factorization $u = u' \circ \phi$ just mentioned. In fact, it may happen that the number of points in the preimage $u^{-1}(p)$, $p \in \text{image}\, u$ is not constant, even away from the set of zeros of du. The reason for that is roughly speaking that points in the interior $z \in \text{Int}\, D$ might be mapped by u to L.

The main tool which enables to deal with this difficulty has been obtained by Lazzarini and, independently, by Kwon and Oh. The key point is the following. Roughly speaking, when a J-holomorphic disk $u : (D, \partial D) \to (M, L)$ is not simple it is possible to decompose its domain D into subdomains $\mathfrak{D}_i$ such that the restriction of u to the closure of each of them, $u|_{\overline{\mathfrak{D}}_i}$, factors through a simple J-holomorphic disk $v_i : (D, \partial D) \to (M, L)$ via a branched covering $\overline{\mathfrak{D}}_i \to D$ of some degree m_i. Moreover, the total homology class is preserved: $[u] = \sum_i m_i [v_i] \in H_2^D(M, L)$. We refer the reader to Lazzarini [28, 27] and to [26] for the precise details.

Coming back to our situation, we know that for generic J the subspace $\mathcal{P}^*_{\mathcal{S}}(I, \mathcal{F}, J) \subset \mathcal{P}_{\mathcal{S}}(I, \mathcal{F}, J)$ formed by elements containing only simple disks are smooth manifolds of the expected dimension. It is therefore enough to show that for generic J, whenever the virtual dimension $\delta_{\mathcal{S}}(I)$ is ≤ 1 we actually have: $\mathcal{P}^*_{\mathcal{S}}(I, \mathcal{F}, J) = \mathcal{P}_{\mathcal{S}}(I, \mathcal{F}, J)$, i.e. all the

disks u participating in elements of the moduli space $\mathcal{P}_{\mathcal{S}}(I, \mathcal{F}, J)$ are simple. This is typically proved using the decomposition technique as follows. Assume for simplicity that $\mathcal{S} = \text{prl}$, $I = (x, y; A)$, $\mathcal{F} = (f, \rho)$. Suppose by contradiction that one of the disks u participating in a pearly trajectory $\mathbf{w} = (w_1, \ldots, w_l) \in \mathcal{P}_{\text{prl}}(x, y; A; f, \rho, J)$ is not simple. We first decompose u - in the sense above - into simple disks, $v_1, \ldots, v_m$. Then, it is possible to find among the v_i's a chain of disks, say $v_{i_0}, v_{i_1}, \ldots$, such that if we replace in $\mathbf{w}$ the disk u by this chain we still get a pearly trajectory $\mathbf{w}' \in \mathcal{P}_{\text{prl}}(x, y; A'; f, \rho, J)$ connecting x to y. Without loss of generality assume that all the disks in $\mathbf{w}'$ are now simple (otherwise we repeat the same procedure). By monotonicity, it follows that the total Maslov index decreases by at least 2, i.e.

$$\mu(A') \leq \mu(A) - N_L \leq \mu(A) - 2.$$

It follows that the virtual dimension of $\mathcal{P}_{\text{prl}}(x, y; A'; f, \rho, J)$ becomes negative:

$$\delta_{\text{prl}}(x, y; A') \leq \delta_{\text{prl}}(x, y; A) - 2 \leq 1 - 2 < 0.$$

By transversality (this time for simple disks) it follows that $\mathcal{P}_{\text{prl}}(x, y; A'; f, \rho, J) = \emptyset$, a contradiction. Thus all elements $\mathbf{w} \in \mathcal{P}_{\text{prl}}(x, y; A; f, \rho, J)$ consist of simple disks.

Similar arguments work also for the other types of moduli spaces $\mathcal{P}_{\text{prod}}$, $\mathcal{P}_{\text{mod}}$ and $\mathcal{P}_{\text{inc}}$. The main difference with respect to $\mathcal{P}_{\text{prl}}$ is that now some of the J-holomorphic disks involved in these spaces have more marked points. For example, in the case of $\mathcal{P}_{\text{mod}}$ one of the disks has -1, 0, 1 as marked points. When applying the preceding argument to such a disk u we do not have good control on how the corresponding marked points are distributed among the v_i's. Nevertheless by a combinatorially more involved argument it is still possible to apply the previous procedure in order to show that $\mathcal{P}^*_{\mathcal{S}}(I, \mathcal{F}, J) = \mathcal{P}_{\mathcal{S}}(I, \mathcal{F}, J)$ for $\mathcal{S} =$ "prod", "mod" and "inc", whenever $\delta_{\mathcal{S}}(I) \leq 1$, hence obtain transversality. The only exceptional case is $S = \text{mod}$, $N_L = 2$, $\delta_S(I) = 1$ which needs to be treated by other methods: roughly, the reason is that $\mathcal{P}_{\text{mod}}$ consists of configurations containing an interior marked point and the reduction to simple disks might increase the degree of liberty of this point so that, as a consequence, the dimension of the respective moduli spaces might not drop by 2 but just by 1.

There is yet another source of complications. Transversality for moduli spaces whose elements involve a single disk at a time can be obtained by restricting to simple disks, but, when considering sequences of pseudo-holomorphic disks $\mathbf{w} = (u_1, \ldots, u_l) \in \mathcal{P}_{\mathcal{S}}(I, \mathcal{F}, J)$ and various evaluation maps, one has to add the assumption that the disks $u_1, \ldots, u_l$ are *absolutely distinct*. This means that for every $1 \leq i \leq l$ we have $u_i(D) \not\subset \cup_{j \neq i} u_j(D)$. It turns out that when the virtual dimension $\delta_{\mathcal{S}}$ is ≤ 1, for generic J all elements of the moduli spaces $\mathcal{P}_{\mathcal{S}}$ indeed consist of absolutely distinct disks. This is also proved using

the monotonicity assumption by similar arguments as above. If the disks in a sequences $\mathbf{w} \in \mathcal{P}_{\mathcal{S}}(I, \mathcal{F}, J)$ are not absolutely distinct then after a suitable omission of some of them we still get an element $\mathbf{w}' \in \mathcal{P}^*_{\mathcal{S}}(I', \mathcal{F}, J)$ in which the disks are absolutely distinct. The point is that by monotonicity, the virtual dimension of $\mathcal{P}^*_{\mathcal{S}}(I', \mathcal{F}, J)$ now becomes negative hence by transversality the latter space is empty. A contradiction.

We refer the reader to [12, 13] for more information and precise details on transversality in the context of pearly moduli spaces.

3.2. **Compactness and gluing.** These are standard ingredients in Morse and Floer theory. In essence, compactness and gluing give a precise description of the boundary of the moduli spaces $\mathcal{P}_{\mathcal{S}}(I, \mathcal{F}, J)$.

For simplicity we elaborate on the case $S = \text{prl}$. A similar discussion applies for the spaces $\mathcal{P}_{\text{prod}}, \mathcal{P}_{\text{mod}}, \mathcal{P}_{\text{inc}}$. Here is a description of the boundary of the space $\mathcal{P}_{\text{prl}}(x, y; A; f, \rho, J)$. Below we abbreviate $\mathcal{F} = (f, \rho)$. Let $\mathbf{w}_k = (u_{1,k}, \ldots, u_{l,k})$ be a sequence in $\mathcal{P}_{\text{prl}}(x, y; A; \mathcal{F}, J)$ that does not have a converging subsequence in that space. Then, after passing to a subsequence, still denoted $\mathbf{w}_k$, we have the following possibilities:

(C-1) One of the gradient trajectories of f breaks at a new critical point $z \in \text{Crit}(f)$, i.e. $\mathbf{w}_k$ splits, as $k \to \infty$, into $\mathbf{w}', \mathbf{w}''$ where $\mathbf{w}' \in \mathcal{P}_{\text{prl}}(x, z; B; \mathcal{F}, J)$, $\mathbf{w}'' \in \mathcal{P}_{\text{prl}}(z, y; C, \mathcal{F}, J)$ and $B + C = A$.

(C-2) One of the gradient trajectories of f connecting adjacent disks, say $u_{i,k}$ to $u_{i+1,k}$, shrinks to a point, i.e. $\mathbf{w}_k$ converges to $(\mathbf{w}', \mathbf{w}'')$, where $\mathbf{w}' = (u'_1, \ldots, u'_{l'}) \in \mathcal{P}_{\text{prl}}(x, p; B; \mathcal{F}, J)$, $\mathbf{w}'' = (u''_1, \ldots, u''_{l''}) \in \mathcal{P}_{\text{prl}}(p, y; C, \mathcal{F}, J)$, $l' \geq 1$, $l'' \geq 1$, $l' + l'' = l$ and $p = u'_{l'}(1) = u''_1(-1)$ is (in general) not a critical point of f. See the left-hand side of Figure 5. Denote by $\mathcal{P}_{\text{prl,C-2}}(x, y; (B, C); \mathcal{F}, J)$ the space of such pairs $(\mathbf{w}', \mathbf{w}'')$ (after moding out by the obvious symmetries coming from reparametrizations of the disks). A simple computation shows that the virtual dimension of this space is:
$$\delta_{\text{prl,C-2}}(x, y; B, C) = |x| - |y| + \mu(B) + \mu(C) - 2.$$

(C-3) Bubbling of a J-holomorphic disk occurs, i.e. there exists $1 \leq i \leq l$ such that the sequence $u_{i,k}$ converges to a reducible J-holomorphic curve consisting of two J-holomorphic disks $u_{i,\infty}$ and $u'_{i,\infty}$ attached to each other at a point on the boundary ∂D. Note that, apriori there are two possibilities for this attaching point. It may be either $\pm 1 \in \partial D$ (i.e. coincide with one of the marked points for elements of $\mathcal{P}_{\text{prl}}$), or it may be another point $\tau \in \partial D \setminus \{-1, 1\}$. The latter case is called *side bubbling*. See the righthand side of Figure 5. In that case we can remove $u'_{i,\infty}$ from the limit

and obtain a new pearly trajectory $\mathbf{w}$ connecting x to y whose total homology class is $A - [u'_{i,\infty}]$. Note that $\mathbf{w}$ belongs to a space whose virtual dimension is

$$\delta_{\mathrm{prl}}(x, y; A - [u'_{i,\infty}]) = \delta_{\mathrm{prl}}(x, y; A) - \mu([u'_{i,\infty}]) \leq \delta_{\mathrm{prl}}(x, y; A) - 2.$$

In the former case (i.e. bubbling occurs at $\tau = \pm 1 \in \partial D$) the limit can be described as a pair $(\mathbf{w}', \mathbf{w}'')$, whose total length is $l+1$, with the same description as elements of $\mathcal{P}_{\mathrm{prl,C\text{-}2}}(x, y; (B, C); \mathcal{F}, J)$. We denote by $\mathcal{P}_{\mathrm{prl,C\text{-}3'}}(x, y; (B, C); \mathcal{F}, J)$ the space of such elements $(\mathbf{w}', \mathbf{w}'')$, where B, C stand for the homology class of the sum of disks in $\mathbf{w}'$ and $\mathbf{w}''$ respectively. Although formally the space $\mathcal{P}_{\mathrm{prl,C\text{-}3'}}(x, y; (B, C); \mathcal{F}, J)$ is the same as $\mathcal{P}_{\mathrm{prl,C\text{-}2}}(x, y; (B, C); \mathcal{F}, J)$ we denote these two spaces differently, since the analytic reason for $\mathbf{w}_k$ converging to a point in each of them is different.

(C-4) Bubbling of a J-holomorphic sphere occurs in one of the disks $u_{i,k}$ of $\mathbf{w}_k$, either at an interior point or at a point on the boundary. If we denote by C the class of the bubbled sphere, then, after removing this sphere from the limit, we obtain a pearly trajectory $\mathbf{w}$ connecting x to y and of total class $A - C$. Thus $\mathbf{w}$ belongs to a space whose virtual dimension is

$$\delta_{\mathrm{prl}}(x, y; A - C) = \delta_{\mathrm{prl}}(x, y; A) - \mu(C) \leq \delta_{\mathrm{prl}}(x, y; A) - 2.$$

(C-5) A combination of (C-1)–(C-4) above, where each of these possibilities can occur repeatedly.

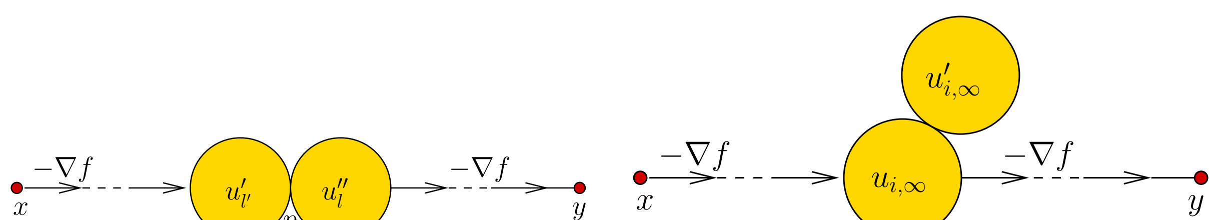

FIGURE 5. On the left: a gradient trajectory has shrunk to a point. On the right: side bubbling.

Continuing with $\mathcal{S} = \mathrm{prl}$, assume now that $\delta_{\mathrm{prl}}(x, y; A) = 0$. We claim that for generic J the space $\mathcal{P}_{\mathrm{prl}}(x, y; A; \mathcal{F}, J)$ is compact. To prove this we notice that in each of the cases (C-1)–(C-5) above the virtual dimension is smaller than $\delta_{\mathrm{prl}}(x, y; A)$ by at least 1 hence negative. A transversality argument, similar to the ones in §3.1, but this time for the spaces $\mathcal{P}_{\mathrm{prl,C\text{-}2}}$, $\mathcal{P}_{\mathrm{prl,C\text{-}3'}}$, shows that when their virtual dimension is ≤ 1 then for generic J these spaces are smooth manifolds of the expected dimension. In our case, since the

virtual dimension is negative, this means that they are just empty sets. As none of the possibilities (C-1)–(C-5) can occur, the space $\mathcal{P}_{\mathrm{prl}}(x, y; A; \mathcal{F}, J)$ is compact (hence a finite set), as claimed by Proposition 3.1.2 and used in formula (3).

Assume now that $\delta_{\mathrm{prl}}(x, y; A) = 1$. We claim that the 1-dimensional manifold $\mathcal{P}_{\mathrm{prl}}(x, y; A; \mathcal{F}, J)$ can be compactified into a manifold with boundary $\overline{\mathcal{P}}_{\mathrm{prl}}$, whose boundary is:

$$\begin{aligned}\partial\overline{\mathcal{P}}_{\mathrm{prl}} = & \bigcup_{\substack{z \in \mathrm{Crit}(f), B+C=A \\ \delta_{\mathrm{prl}}(x,z;B)=0 \\ \delta_{\mathrm{prl}}(z,y;C)=0}} \mathcal{P}_{\mathrm{prl}}(x, z; B; \mathcal{F}, J) \times \mathcal{P}_{\mathrm{prl}}(z, y; C; \mathcal{F}, J) \coprod \\ & \bigcup_{B+C=A} \mathcal{P}_{\mathrm{prl,C\text{-}2}}(x, y; B, C; \mathcal{F}, J) \coprod \bigcup_{B+C=A} \mathcal{P}_{\mathrm{prl,C\text{-}3'}}(x, y; B, C; \mathcal{F}, J).\end{aligned} \tag{13}$$

To see this, first note that elements corresponding to possibility (C-4) cannot occur in the boundary of $\mathcal{P}_{\mathrm{prl}}(x, y; A; \mathcal{F}, J)$ when $\delta_{\mathrm{prl}}(x, y; A) = 1$. The reason is that these elements correspond to spaces of pearly trajectories whose virtual dimension in (C-4) is $\leq \delta_{\mathrm{prl}}(x, y; A) - 2 < 0$. Thus by transversality for pearls these spaces are empty. For a similar reason side bubbling cannot occur either. Thus, we are left with possibilities (C-1), (C-2) and (C-3'). This shows that $\partial\overline{\mathcal{P}}_{\mathrm{prl}} \subset$ (RHS of (13)).

It remains to show that the opposite inclusion $\partial\overline{\mathcal{P}}_{\mathrm{prl}} \supset$ (RHS of (13)) holds too. This is a consequence of the gluing procedure. The precise details of gluing are beyond the scope of this paper and we skip the details. The fact that $\overline{\mathcal{P}}_{\mathrm{prl}}$ as described in (13) has the structure of a 1-dimensional manifold with boundary, i.e. that each element on the RHS of (13) corresponds to a *unique* end of the (possibly non-compact) manifold $\mathcal{P}_{\mathrm{prl}}$ is a consequence of the so called surjectivity of the gluing map.

Here is some reference on the gluing procedure. Gluing of closed pseudo-holomorphic curves is presented in a very detailed way in [29]. Gluing of pseudo-holomorphic disks is developed in [23] and further elaborated in [12] where is also treated the surjectivity of the gluing map.

3.3. **Putting everything together and the main scheme of proof.** We are now ready to prove that $d \circ d = 0$ as claimed by Theorem 2.3.1. This final step is standard in Morse-Floer theory and goes as follows. Fix $x \in \mathrm{Crit}(f)$. For every $y \in \mathrm{Crit}(f)$ denote

by $\langle d\circ d(x), y\rangle$ the coefficient (in Λ) of y in $d\circ d(x)\in\mathcal{C}(f,\rho,J)$. By definition:

$$\begin{aligned}\langle d\circ d(x), y\rangle &= \sum_{\substack{z\in\mathrm{Crit}(f),B,C\\ \delta_{\mathrm{prl}}(x,z,B)=0\\ \delta_{\mathrm{prl}}(z,y,C)=0}} \#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}(x,z;B;\mathcal{F},J)\#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}(z,y;C;\mathcal{F},J)t^{\bar\mu(B)+\bar\mu(C)} = \\ & \left(\sum_{\substack{z\in\mathrm{Crit}(f),B,C\\ \mu(B+C)=2-|x|+|y|\\ \delta_{\mathrm{prl}}(x,z,B)=0}} \#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}(x,z;B;\mathcal{F},J)\#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}(z,y;C;\mathcal{F},J)\right)t^{(2-|x|+|y|)/N_L}.\end{aligned} \tag{14}$$

The last equality here follows from the fact that $\delta_{\mathrm{prl}}(x,z,B)=\delta_{\mathrm{prl}}(z,y,C)=0$ iff $\mu(B)+\mu(C)=2-|x|+|y|$ and $\delta_{\mathrm{prl}}(x,z;B)=0$. Thus the factor $t^{\bar\mu(B)+\bar\mu(C)}$ is constant and always equals $t^{(2-|x|+|y|)/N_L}$.

To prove that the sum in (14) is 0 we use the description (13) of the boundary of $\overline{\mathcal{P}}_{\mathrm{prl}}$. Fix $A\in H_2^D(M,L)$ with $\mu(A)=2-|x|+|y|$. Since $\overline{\mathcal{P}}_{\mathrm{prl}}(x,y;A;\mathcal{F},J)$ is a compact 1-dimensional manifold with boundary, its boundary consists of an even number of points. Thus, by (13) we have:

$$\begin{aligned} 0 &= \#_{\mathbb{Z}_2}\partial\overline{\mathcal{P}}_{\mathrm{prl}}(x,y;A;\mathcal{F},J) = \\ & \sum_{\substack{z\in\mathrm{Crit}(f),B+C=A\\ \delta_{\mathrm{prl}}(x,z;B)=0\\ \delta_{\mathrm{prl}}(z,y;C)=0}} \#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}(x,z;B;\mathcal{F},J)\#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}(z,y;C;\mathcal{F},J) \ + \\ & \sum_{B+C=A}\#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl,C\text{-}2}}(x,y;B,C;\mathcal{F},J) \ + \sum_{B+C=A}\#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl,C\text{-}3'}}(x,y;B,C;\mathcal{F},J).\end{aligned}\tag{15}$$

However, as noted in §3.2 above (see case (C-3) there) the spaces $\mathcal{P}_{\mathrm{prl,C\text{-}2}}(x,y;B,C;\mathcal{F},J)$ and $\mathcal{P}_{\mathrm{prl,C\text{-}3'}}(x,y;B,C;\mathcal{F},J)$ are actually two identical copies of the same space. Thus the sum on the last line of (15) vanishes (in $\mathbb{Z}_2$). Summing now equality (15) over all possible classes A with $\mu(A)=2-|x|+|y|$ we obtain the needed equality. This concludes the (outline of the) proof that $d\circ d=0$. □

3.4. **The identification maps.** As in Morse theory, there are essentially two techniques to construct a comparison chain morphism

$$\psi_{\mathscr{D}',\mathscr{D}}:\mathcal{C}_*(\mathscr{D})\longrightarrow\mathcal{C}_*(\mathscr{D}')$$

for every two generic triples $\mathscr{D}=(f,\rho,J)$, $\mathscr{D}'=(f',\rho',J')$.

The first method is based on using Morse cobordisms. Such a cobordism is a pair $(F,\bar\rho)$ defined on the product: $F:L\times[0,1]\to\mathbb{R}$ and $\bar\rho$ a metric on $L\times[0,1]$ and so that (up to the possible addition of an appropriate constant) we have $(F,\bar\rho)|_{L\times\{0\}}=(f,\rho)$ and $(F,\bar\rho)|_{L\times\{1\}}=(f',\rho')$; the pair $(F,\bar\rho)$ is Morse-Smale and $\mathrm{Crit_i(F)}=\mathrm{Crit_{i-1}(f)}\times\{0\}\cup$

$\mathrm{Crit_i(f)} \times \{1\}$; $\frac{\partial F}{\partial t}(x,t) = 0$ for $(x,t) \in L \times \{0,1\}$ and $\frac{\partial F}{\partial t}(x,t) < 0$ if $t \in L \times (0,1)$. We also consider a smooth one parametric family of ω-compatible almost complex structures $\bar{J}_t$ so that $\bar{J}_0 = J$ and $\bar{J}_1 = J'$. We then define pearl type moduli spaces as in §2.3 but with a couple of modifications: the place of the flow Φ is now taken by the flow $\bar{\Phi}$ on $L \times [0,1]$ induced by $-\nabla_{\bar{\rho}} F$; the non-constant disks u_i are $\bar{J}_{\tau i}$-holomorphic where, as in the definition of $\mathcal{P}_{\mathrm{prl}}$, t_i is so that $\bar{\Phi}_{t_i}(u_i(+1)) = u_{i+1}(-1)$ and the parameter τ_i is determined by $\tau_i = \mathrm{pr}_2(\phi_{t_i}(u_i(+1)))$ with $\mathrm{pr}_2 : L \times [0,1] \to [0,1]$ the projection on the second factor. The transversality issues for these moduli spaces are perfectly similar to those for the usual pearl moduli spaces. Under generic choices for $F, \bar{\rho}, \bar{J}$ counting (mod 2) the elements in the 0-dimensional such moduli spaces defines the chain morphism $\psi_{\mathscr{D}',\mathscr{D}}$ as desired. The same construction is then applied to cobordisms of Morse cobordisms and it shows that the induced map in homology, $\Psi_{\mathscr{D}',\mathscr{D}}$, is canonical.

The second method is more direct. Given the two data sets $\mathscr{D} = (f, \rho, J)$, $\mathscr{D}' = (f', \rho', J')$ we consider moduli spaces consisting of triples $(\mathbf{u}, \mathbf{v}, p)$ with $\mathbf{u} \in \mathcal{P}_{\mathrm{prl}}(x, p; A; f, \rho, J)$, $\mathbf{v} \in \mathcal{P}_{\mathrm{prl}}(p, y'; A'; f', \rho', J')$, $p \in L$. It is easily seen that counting 0-dimensional such configurations gives a chain morphism:

$$\psi'_{\mathscr{D}',\mathscr{D}} : \mathcal{C}_*(\mathscr{D}) \longrightarrow \mathcal{C}_*(\mathscr{D}') \ .$$

The disadvantage of this second method is that, in this case, it is harder to directly check that the map induced in homology, $\Psi'_{\mathscr{D}',\mathscr{D}}$, is canonical. Moreover, for the moduli spaces involved in the definition of the morphism ψ' to be regular, the two pairs (f, ρ) and (f', ρ') need to be generic in the sense that the unstable manifolds of f are required to be transverse to the stable manifolds of f'.

However, it is not difficult to verify that $\Psi'_{\mathscr{D}',\mathscr{D}} = \Psi_{\mathscr{D}',\mathscr{D}}$. Thus, both methods produce the same (canonical) morphism in homology.

3.5. **Proving Theorems 2.4.1- 2.6.1.** As mentioned earlier, the proofs of Theorems 2.4.1-2.6.1 follow the same scheme as the proof of $d^2 = 0$ for the differential of the pearl complex. For example, in order to show that each of the maps $\circ$, $\circledast$ and i_L are chain maps we compactify 1-dimensional spaces of the type $\mathcal{P}_S(I, \mathcal{F}, J)$ into compact 1-dimensional manifolds with boundary. To prove the other statements such as the associativity of $\circ$ in homology, the fact that $\circledast$ is indeed a module operation etc. we follow the same scheme, but now we have to work with other types of moduli spaces that haven't been introduced explicitly above. The main difference is that some of the pseudo-holomorphic disks will have now more marked points and, possibly, there will be more than a single disk with several marked points. We refer the reader to [12, 13] for the precise details. Transversality in these cases as well as in the exceptional case $\mathcal{S} = \mathrm{mod}$, $N_L = 2, \delta_{\mathcal{S}}(I) = 1$ can be achieved

by the scheme described before only after allowing that some of the curves in the chain of pearls configurations carry Hamiltonian perturbations of the type described in [1].

There is a unified approach to all these issues which is based on trees. More precisely we consider planar oriented trees whose edges and vertices are labeled as follows. The edges are labeled by Morse functions (some on L, some on M) and the inner vertices by elements of $H_2^D(M, L)$. The entries and exit vertices are labeled by critical points of the function corresponding to the adjacent edge.

Each such tree (or collection of trees) determines in a natural way a moduli space involving gradient trajectories attached to pseudo-holomorphic disks. An appropriate count of the number of elements in the 0-dimensional components of these spaces gives rise to a quantum operation on the chain level. For example, the pearly differential is defined by looking at linear trees (i.e. one entry and one exit). The quantum product is defined by considering trees with two entries and one exit all having valence 1.

The advantage in modeling all the moduli spaces on trees is that most of the arguments involving compactness, gluing and transversality can be proved for large classes of trees and there is no need to repeat small variations of each argument over and over again for each quantum operation separately. This is particularly useful in dealing with the moduli spaces that appear in the proof of the various associativity relations involving the quantum product and the module structure as well as to keep track of the Hamiltonian perturbations which are required. In [13] this approach is described in full, for all the moduli spaces needed for these operations as well as for the relations among them.

The idea to model homological operations in Morse and Floer theory on graphs is not new and has been implemented in various settings, see e.g. [6] for the Morse case, [23] for Lagrangian Floer theory as well as [18] (where the point of view is closest to that of the present paper).

3.6. **Identification with Floer homology.** The version of Floer homology that we need is defined in the presence of a Hamiltonian $H : M \times [0, 1] \to \mathbb{R}$. Consider the path space $\mathcal{P}_0(L) = \{\gamma \in C^\infty([0, 1], M) \mid \gamma(0) \in L\ ,\ \gamma(1) \in L\ ,\ [\gamma] = 0 \in \pi_2(M, L)\}$ and inside it the set of contractible orbits $\mathcal{O}_H \subset \mathcal{P}_0(L)$ of the Hamiltonian flow X_H. Assuming H to be generic we have that $\mathcal{O}_H$ is a finite set. Fix some almost complex structure J. The Maslov index induces a morphism $\mu : \pi_1\mathcal{P}_0(L) \to \mathbb{Z}$ and we let $\tilde{\mathcal{P}}_0(L)$ be the regular, abelian cover associated to $\ker(\mu)$, the group of deck transformations being $\pi_1(\mathcal{P}_0(L))/\ker(\mu)$. Consider all the lifts $\tilde{x} \in \tilde{\mathcal{P}}_0(L)$ of the orbits $x \in \mathcal{O}_H$ and let $\tilde{\mathcal{O}}_H$ be the set of these lifts. Fix a basepoint η_0 in $\tilde{\mathcal{P}}_0(L)$ and define the degree of each element $\tilde{x}$ by $|\tilde{x}| = \mu(\tilde{x}, \eta_0)$ with μ being here the Viterbo-Maslov index. The Floer complex is the Λ-module:

$$CF_*(H, J) = \mathbb{Z}_2\langle\tilde{\mathcal{O}}_H\rangle$$

where $t^r \in \Lambda$ acts on $\tilde{x}$ by $t^r\tilde{x} = rN_L \cdot \tilde{x}$. The differential is given by $dx = \sum \#\mathcal{M}(\tilde{x}, \tilde{y})\tilde{y}$ where $\mathcal{M}(\tilde{x}, \tilde{y})$ is the moduli space of solutions $u : \mathbb{R} \times [0, 1] \to M$ of Floer's equation $\partial u/\partial s + J\, \partial u/\partial t + \nabla H(u, t) = 0$ which verify $u(\mathbb{R} \times \{0\}) \subset L,\ u(\mathbb{R} \times \{1\}) \subset L$ and they lift in $\tilde{\mathcal{P}}_0(L)$ to paths relating $\tilde{x}$ and $\tilde{y}$. Moreover, the sum is subject to the condition $\mu(\tilde{x}, \tilde{y}) - 1 = 0$.

The comparison map from the pearl complex

$$\Phi_{f,H} : \mathcal{C}(f, \rho, J) \to CF(L; H, J)$$

is defined by the PSS method (see [34] and, in the Lagrangian case, [5], [18],[2]) as well as the map in the opposite direction

$$\Theta_{H,f} : CF(H, J) \to \mathcal{C}(f, \rho, J)\ .$$

For example, the value of the map $\Phi_{f,H}$ on the generator $x \in \mathrm{Crit(f)}$ is defined by counting elements in (0-dimensional) moduli spaces consisting of triples $(\mathbf{u}, p, v)$ so that $p \in L$, $\mathbf{u} \in \mathcal{P}_{\mathrm{prl}}(x, p; f, \rho, J)$ and v is a solution of the equation

$$\partial v/\partial s + J\partial v/\partial t + \beta(s)\nabla H(v, t) = 0 \tag{16}$$

so that $\beta : \mathbb{R} \to [0, 1]$ is an appropriate increasing smooth function supported in the interval $[-1, +\infty)$ and which is constant equal to 1 on $[1, +\infty)$. This solution v has also to verify $v(\mathbb{R} \times \{0\}) \subset L$, $v(\mathbb{R} \times \{1\}) \subset L$, $\lim_{s\to\infty} v(s, -) = \gamma(-)$ and $\lim_{s\to-\infty} v(s, -) = p \in L$. Transversality issues can be dealt with by methods similar to those described in the case of the pearl complex.

The value of the map Θ on some element $\gamma \in \mathcal{O}_H$ is given by using similar moduli spaces which now consist of triples $(v, p, \mathbf{u})$, with $a \in L$, $\mathbf{u} \in \mathcal{P}_{\mathrm{prl}}(p, y; f, \rho, J)$ and v verifying an equation like (16) but with the function β replaced by $\beta' = 1 - \beta$ and $\lim_{s\to-\infty} v(s, -) = \gamma(-)$ and $\lim_{s\to\infty} v(s, -) = p$. Proving that these maps are chain morphisms and that their compositions induce inverse maps in homology depends, in the first instance, on using one-dimensional moduli spaces as above and, in the second, on yet some other moduli spaces which will produce the needed chain homotopies (see again [12] and [13] as well as [2] for details). It is easy to see that these morphisms identify the module and quantum product as defined in "pearl" terms with the analogue structures defined in Floer homology.

4. Further structures

4.1. **Augmentation, duality and spectral sequences.** There are a number of additional algebraic structures associated to the quantum homology of a monotone Lagrangian L and we review here the most significant of them.

4.1.1. *Augmentation.* Given a pearl complex $\mathcal{C}(f,\rho,J) = \mathbb{Z}_2\langle \mathrm{Crit(f)}\rangle \otimes \Lambda$, define a map:

$$\epsilon_L : \mathcal{C}(f,\rho,J) \to \Lambda$$

by $\epsilon_L(x) = 1$ for all $x \in \mathrm{Crit}_0(\mathrm{f})$ and $\epsilon_L(x) = 0$ for all critical points of f of strictly positive index. It is easy to see that this is a chain map (where the differential on Λ is trivial) and that the map induced in homology - which is called the *augmentation* is canonical.

By using the augmentation it is easy to see that the quantum inclusion is actually determined by the module action. The following formula is true

$$\langle PD(h), i_L(x)\rangle = \epsilon_L(h \circledast x) \tag{17}$$

for all $h \in H_*(M;\mathbb{Z}_2),\ x \in QH(L)$ with $PD(-)$ Poincaré duality and $\langle -.-\rangle$ the Kronecker pairing.

4.1.2. *Duality.* Assuming defined the chain complex $\mathcal{C}(f,\rho,J)$ the dual co-chain complex associated to it is given by

$$\mathcal{C}^*(f,\rho,J) = (\hom_{\mathbb{Z}_2}(\mathbb{Z}_2\langle \mathrm{Crit(f)}\rangle, \mathbb{Z}_2) \otimes \Lambda, \mathrm{d}^*)$$

where if $x \in \mathrm{Crit_i(f)}$, then the degree of $x^* \in \hom_{\mathbb{Z}_2}(\mathbb{Z}_2\langle \mathrm{Crit(f)}\rangle, \mathbb{Z}_2)$, the dual of x, is i; the differential d^* is the dual of d. The co-homology of this complex is again canonical and it computes, by definition, the *quantum co-homology* of L, $QH^*(L)$. Clearly, we have an evaluation $QH^* \otimes QH_* \to \Lambda$ which we write as $\sigma \otimes \alpha \to \sigma(\alpha)$.

Theorem 4.1.1. *There is a canonical isomorphism*

$$\eta : QH_k(L) \to QH^{n-k}(L)$$

which corresponds to the bilinear map: $\bar{\eta} : QH_k(L) \otimes QH_{k'}(L) \xrightarrow{\circ} QH_{k+k'-n}(L) \xrightarrow{\epsilon_L} \Lambda$ *via the relation* $\eta(x)(y) = \bar{\eta}(x \otimes y)$.

The isomorphism η is obtained by composing the standard comparison map $\psi_{-f,f} : \mathcal{C}(f,\rho,J) \to \mathcal{C}(-f,\rho,J)$ (which is defined for generic choices of data f,ρ,J as in §3.4) with the identification of $\mathcal{C}(-f,\rho,J)$ and $\mathcal{C}^*(f,\rho,J)$ induced by $x \to x^*$, $\forall x \in \mathrm{Crit(-f)} = \mathrm{Crit(f)}$. We refer to [12] [13] for full details.

The quantum inclusion, i_L, the duality map, η, and the Lagrangian quantum product determine the module structure by the following formula which extends (17):

$$\langle PD(h), i_L(x \circ y)\rangle = \langle \eta(y), h \circledast x\rangle \tag{18}$$

where $h \in H_*(M;\mathbb{Z}_2)$, $x, y \in QH(L)$.

4.1.3. *Degree filtration and the associated spectral sequence.* All the structures discussed in this paper are based on moduli spaces which consist of configurations consisting of pseudo-holomorphic objects joined together by Morse trajectories. In particular, all these objects have a positive symplectic area and, if this area is null, then they reduce to the classical Morse moduli spaces associated to the structure in question.

As a consequence, all these structures respect the filtration of Λ by the degrees of t:

$$\Lambda^k = t^k \mathbb{Z}_2[t] \ .$$

In particular, the pearl complex $\mathcal{C}(f, \rho, J)$ is filtered by

$$F^k \mathcal{C}(f, \rho, J) = \mathbb{Z}_2 < \mathrm{Crit(f)} > \otimes \Lambda^{\mathrm{k}}$$

and the pearl differential respects this filtration. Thus there is a spectral sequence associated to this filtration which converges to the quantum homology of L and whose term (E^0, d^0) is just the Morse complex of f (tensored with Λ). This spectral sequence is a variant of the spectral sequence introduced by Oh in [32]. The quantum product as well as the module action also respect this filtration.

4.2. **Other coefficient rings.** Here we extend the quantum homology $QH(L)$ to larger coefficient rings which also take into account the actual homology classes of the pearly trajectories, not only their total Maslov index. As mentioned above, since we are in the monotone case we can actually work with rings taking into account the positivity of the Maslov index of pseudo-holomorphic curves. Indeed, all our operations, differentials and comparison maps - with the notable exception of the identification map with Floer homology - only involve holomorphic objects and so only involve classes for which the Maslov class is positive. The resulting quantum homology of L carries more information than $QH(L)$ as defined in §2.3.

Let $H_2^S(M, L) \subset H_2(M; \mathbb{Z})$ be the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M; \mathbb{Z})$, and $H_2^S(M)^+ \subset H_2^S(M)$ the semi-group consisting of classes A with $c_1(A) > 0$. Similarly, denote by $H_2^D(M, L)^+ \subset H_2^D(M, L)$ the semi-group of elements A with $\mu(A) > 0$. Let $\widetilde{\Gamma}^+ = \mathbb{Z}_2[H_2^S(M)^+] \cup \{1\}$ be the unitary ring obtained by adjoining a unit to the non-unitary group ring $\mathbb{Z}_2[H_2^S(M)^+]$. Similarly we put $\widetilde{\Lambda}^+ = \mathbb{Z}_2[H_2^D(M, L)^+] \cup \{1\}$. We write elements of $Q \in \widetilde{\Gamma}^+$ and $P \in \widetilde{\Lambda}^+$ as "polynomials" in the formal variable S and T:

$$Q(S) = a_0 + \sum_{c_1(A)>0} a_A S^A, \qquad P(T) = b_0 + \sum_{\mu(B)>0} b_B T^B \qquad a_0, a_A, b_0, b_B \in \mathbb{Z}_2.$$

We endow these rings with the following grading:

$$deg S^A = -2c_1(A), \quad \deg T^B = -\mu(B).$$

Note that these rings are smaller than the rings $\hat{\Gamma}^{\geq 0} = \mathbb{Z}_2[\{A | c_1(A) \geq 0\}]$ and $\hat{\Lambda}^{\geq 0} = \mathbb{Z}_2[\{B | \mu(B) \geq 0\}]$. For example, $\hat{\Lambda}^{\geq 0}$ and $\hat{\Gamma}^{\geq 0}$ might have many non-trivial elements in degree 0, whereas in $\widetilde{\Gamma}^+$ and $\widetilde{\Lambda}^+$ the only such element is 1.

Let $QH(M; \widetilde{\Gamma}^+)$ be the quantum homology of M with coefficients in $\widetilde{\Gamma}^+$ endowed with the quantum product $*$. We have a natural map $H_2^S(M)^+ \to H_2^D(M, L)^+$ which induces on $\widetilde{\Lambda}^+$ a structure of a $\widetilde{\Gamma}^+$-module. Put $QH(M; \widetilde{\Lambda}^+) = QH(M; \widetilde{\Gamma}^+) \otimes_{\widetilde{\Gamma}^+} \widetilde{\Lambda}^+$ and endow it with the quantum intersection product, still denoted $*$ (defined e.g. as in [29]). Note that the quantum product is well defined with this choice of coefficients, since by monotonicity the only possible pseudo-holomorphic sphere with Chern number 0 is constant. We grade this ring with the obvious grading coming from the two factors.

Given a triple $\mathscr{D} = (f, \rho, J)$ put $\mathcal{C}(\mathscr{D}; \widetilde{\Lambda}^+) = \mathbb{Z}_2\langle \mathrm{Crit}(f) \rangle \otimes \widetilde{\Lambda}^+$ endowed with the grading coming form both factors. We define a map $\widetilde{d}^+ : \mathcal{C}_*(\mathscr{D}; \widetilde{\Lambda}^+) \longrightarrow \mathcal{C}_{*-1}(\mathscr{D}; \widetilde{\Lambda}^+)$ by changing the differential d in formula (3) as follows: instead of the coefficient $t^{\bar{\mu}(A)}$ put T^A for $\widetilde{d}^+$. Note that $\widetilde{d}^+$ is well defined due to monotonicity. Indeed, if $\mathbf{u}$ is a pearly trajectory with total homology class A then either $A = 0$ or $\mu(A) > 0$. Therefore $T^A \in \widetilde{\Lambda}^+$.

We alter all the other operations, $\circ$, $\circledast$ and i_L described in §2 by rewriting all formulas with the coefficient ring $\widetilde{\Lambda}^+$.

Theorem 4.2.1. *The map $\widetilde{d}^+$ is a differential and the homology of $\mathcal{C}_*(\mathscr{D}; \widetilde{\Lambda}^+, \widetilde{d}^+)$ denoted $QH_*(L; \widetilde{\Lambda}^+)$ is independent of the choice of the generic triple $\mathscr{D} = (f, \rho, J)$. Furthermore, all the statements in Theorems 2.3.1- 2.6.1, except of the comparison Θ with $HF(L, L)$, continue to hold when replacing $QH(L)$ by $QH(L; \widetilde{\Lambda}^+)$ and $QH(M)$ by $QH(M; \widetilde{\Lambda}^+)$.*

The proof of this theorem is essentially the same as the proofs of Theorems 2.3.1- 2.6.1. The main point is that, not only the total Maslov index, but also the total homology class is preserved under bubbling as well as under gluing.

Let $\mathcal{R}$ be a commutative $\widetilde{\Lambda}^+$-algebra. Consider the complex

$$\mathcal{C}(\mathscr{D}; \mathcal{R}) = \mathcal{C}(\mathscr{D}; \widetilde{\Lambda}^+) \otimes_{\widetilde{\Lambda}^+} \mathcal{R}$$

endowed with the differential $d^{\mathcal{R}}$ induced from $\widetilde{d}^+$. We denote the homology of this complex by $QH_*(L; \mathcal{R})$. Finally we extend the coefficients of the quantum homology of the ambient manifold by $QH(M; \mathcal{R}) = QH(M; \widetilde{\Lambda}^+) \otimes_{\widetilde{\Lambda}^+} \mathcal{R}$. Clearly, the statement of Theorem 4.2.1 continues to hold when replacing $\widetilde{\Lambda}^+$ by $\mathcal{R}$. Moreover, there exists a canonical map $QH_*(L; \widetilde{\Lambda}^+) \to QH_*(L; \mathcal{R})$ induced by the obvious ring homomorphism $\widetilde{\Lambda}^+ \to \mathcal{R}$.

Here are a few examples of rings $\mathcal{R}$ which are useful in applications. We endow a commutative ring $\mathcal{R}$ with the structure of $\widetilde{\Lambda}^+$-algebra by specifying a ring homomorphism $q : \widetilde{\Lambda}^+ \to \mathcal{R}$.

(1) Take $\mathcal{R} = \Lambda = \mathbb{Z}_2[t^{-1}, t]$, and define q by $q(T^A) = t^{\bar{\mu}(A)}$. It is easy to see that $QH(L;\Lambda)$ coincides with our original homology $QH(L)$.
(2) Take $\mathcal{R} = \mathbb{Z}_2[t]$, and define q as in 1. We denote this ring by Λ^+ and the resulting homology $QH(L;\Lambda^+)$ by $Q^+H(L)$.
(3) Take $\mathcal{R} = \mathbb{Z}_2[H_2^D(M,L)]$ with the obvious $\widetilde{\Lambda}^+$-algebra structure.

Remark 4.2.2. i. While $QH(L)$ is isomorphic to $HF(L,L)$ the relation of the homology $QH(L;\widetilde{\Lambda}^+)$ to $HF(L,L)$ is not straightforward. For example, while $HF(L,L)$ might vanish (e.g. when L is displaceable) this is *never* the case for $QH(L;\widetilde{\Lambda}^+)$. To see this recall that if $f : L \to \mathbb{R}$ is a Morse function with a single maximum x then $x \in \mathcal{C}_n(f,\rho,J;\widetilde{\Lambda}^+)$ is a cycle and its homology class $[x]$ is the unity of $QH_*(L;\widetilde{\Lambda}^+)$. Thus $QH_*(L;\widetilde{\Lambda}^+)$ vanishes iff x is a boundary. However, it is easy to see that for degree reasons $\mathcal{C}_{n+1}(f,\rho,J;\widetilde{\Lambda}^+) = 0$, hence x cannot have a $\widetilde{d}^+$-primitive. The same remark applies to $QH(L;\mathcal{R})$ where $\mathcal{R}$ is a $\widetilde{\Lambda}^+$-algebra with no elements of negative degree e.g. $\mathcal{R} = \Lambda^+$.

Let us mention that working with rings such as $\widetilde{\Lambda}^+$ in the context of Floer homology has been considered before in [23] where the relation between displacement energy of a Lagrangian and algebraic properties of the torsion of the corresponding Floer homology is studied.

ii. Let $\widehat{\Lambda} = \mathbb{Z}_2[H_2^D(M,L)]$ and let $\mathcal{R}$ be a $\widehat{\Lambda}$-algebra. Under these assumptions it is possible to define a version of Floer homology $HF(L,L;\mathcal{R})$ over $\mathcal{R}$ in an analogous way to the usual definition (see [23] as well as [12]). On the other hand since $\widehat{\Lambda}$ is a $\widetilde{\Lambda}^+$-algebra so is $\mathcal{R}$ hence we can define also $QH(L;\mathcal{R})$. It turns out that the comparison map Θ (see Theorem 2.3.1) can be extended to this case. It gives a canonical isomorphism (upto a shift in grading) $HF_*(L,L;\mathcal{R}) \cong QH_*(L;\mathcal{R})$.

iii. All the coefficient rings discussed above are based on group rings of commutative groups or semi-groups (in particular, $H_2^D(M,L)^+$). We could also have used directly the semi group $\pi_2(M,L)^+$ which consists of the elements of $\alpha \in \pi_2(M,L)$ so that $\omega(\alpha) > 0$ or, when a group is required, $\pi_2(M,L)$. Both are, in general, non-commutative. For now, we have not used this non-commutative ring in applications and so we have only treated above the more familiar, commutative case.

4.2.1. *An example.* Here is a simple example which illustrate the various types of homologies considered here.

Let $L \subset \mathbb{R}^2$ be an embedded circle. This is a monotone Lagrangian with $N_L = 2$. We have $QH(L) \cong HF(L,L) = 0$ since L is displaceable. Let us compute now $Q^+H(L)$ (i.e. $QH(L;\Lambda^+)$, where $\Lambda^+ = \mathbb{Z}_2[t]$). Let $f : L \longrightarrow \mathbb{R}^2$ be a Morse function with two critical points: the maximum x_1 and the minimum x_0. Let ρ be any Riemannian metric on L

and J any almost complex structure compatible with the symplectic structure of $\mathbb{R}^2$. We have:

$$\mathcal{C}_i(f,\rho,J) = \begin{cases} 0, & i \geq 2 \\ \mathbb{Z}_2 x_1 t^k, & i = 1-2k,\ k \geq 0 \\ \mathbb{Z}_2 x_0 t^k, & i = -2k,\ k \geq 0 \end{cases} \tag{19}$$

As for the differential d^+, we have: $d^+(x_1) = 0$, $d^+(x_0) = x_1 t$. The first equality is because there are exactly two ($= 0 \in \mathbb{Z}_2$) negative gradient trajectories going from x_1 to x_0 and no other pearly trajectories form x_1 to x_0. As for $d^+(x_0)$, it is easy to see that there is precisely one pearly trajectory from x_0 to x_1. This trajectory starts at x_0, then involves the (single) holomorphic disk spanning L and then stops at x_1. This disk has Maslov index 2. It is easy to see that this is the single pearly trajectory from x_0 to x_1. This proves that $d^+(x_0) = x_1 t$.

Passing to homology, we see that $Q^+H_i(L) = 0$ for every $i \neq 1$, and $Q^+H_1(L) \cong \mathbb{Z}_2[x_1]$. Clearly $[x_1]$ is a torsion element in the sense that $t[x_1] = 0$.

4.2.2. *Other ground fields and rings.* All the constructions and results in §2 are very much likely to hold true if we replace the ground field by $\mathbb{Q}$ or even $\mathbb{Z}$ provided that the Lagrangians L are assumed to be orientable and relative spin. Indeed due to [23] under these conditions it is possible to orient in a coherent way the moduli spaces of pseudo-holomorphic disks, hence also the pearly moduli spaces $\mathcal{P}_{\text{prl}}$, $\mathcal{P}_{\text{prod}}$, $\mathcal{P}_{\text{mod}}$ and $\mathcal{P}_{\text{inc}}$. The only thing that remains to be rigorously verified is that these orientations are compatible with the algebraic structures introduced in §2.

4.3. **Relation to classical operations revisited.** The relation of the quantum operations $\circ$, $\circledast$ and i_L to their classical counterparts which was discussed in §2.7 can be further clarified using the ring $\Lambda^+ = \mathbb{Z}_2[t]$. We view this ring as a $\widetilde{\Lambda}^+$-algebra as explained in the preceding section.

Fix a generic triple (f,ρ,J). Note that the pearl complex with coefficients in Λ^+ can be simply written as:

$$\mathcal{C}(f,\rho,J;\Lambda^+) = \mathbb{Z}_2\langle \mathrm{Crit}(f)\rangle \otimes \mathbb{Z}_2[t], \quad d = \partial_0 + \partial_1 t + \cdots + \partial_\nu t^\nu, \tag{20}$$

where the operators ∂_i are as described in §2.7. Recall also that ∂_0 coincides with the Morse-homology differential. Denote by $C(f,\rho) = \mathbb{Z}_2\langle \mathrm{Crit}(f)\rangle$ the Morse complex (endowed with the differential ∂_0).

Consider the specialization homomorphism $\mathbb{Z}_2[t] \to \mathbb{Z}_2$ defined by $t \mapsto 0$. It induces a map $\tilde{\sigma} : \mathcal{C}_*(f,\rho,J;\Lambda^+) \to C_*(f,\rho)$. A simple computation based on (20) shows that $\tilde{\sigma}$ is a chain map hence induces a map in homology $\sigma : Q^+H_*(L) \longrightarrow H_*(L;\mathbb{Z}_2)$. This map can

be viewed as a comparison map between the quantum structures and the classical ones. For example, $\forall \quad \alpha, \beta \in Q^+H(L), a \in QH(M;\Lambda^+)$ we have:

$$\sigma(\alpha \circ \beta) = \sigma(\alpha) \cap \sigma(\beta), \quad \sigma(a \circledast \alpha) = a \widetilde{\cap} \sigma(\alpha), \quad \text{inc}_*(\sigma(\alpha)) = \pi(i_L(\alpha)). \tag{21}$$

Here $\cap$ is the classical intersection product on $H(L;\mathbb{Z}_2)$, $\widetilde{\cap} : H(M;\mathbb{Z}_2) \otimes H(L;\mathbb{Z}_2) \longrightarrow H(L;\mathbb{Z}_2)$ is the exterior intersection product defined by intersecting cycles in M with cycles in L. The map $\text{inc}_* : H_*(L;\mathbb{Z}_2) \longrightarrow H_*(M;\mathbb{Z}_2)$ is the canonical map induced by the inclusion $L \subset M$. Finally, $\pi : QH(M;\Lambda^+) \to H(M;\mathbb{Z}_2)$ is the projection onto the $H(M;\mathbb{Z}_2)$ summand of $QH * M;\Lambda^+)$ corresponding to $t = 0$.

The proof of the identities in (21) follows immediately from the discussion in §2.7.

4.4. **Action estimates.** Given a monotone Lagrangian $L \subset (M, \omega)$ we have described in §2.5 the quantum module structure:

$$QH(M;\Lambda) \circledast QH(L) \to QH(L) \ . \tag{22}$$

For a Hamiltonian $H : M \times S^1 \to \mathbb{R}$ there is a PSS isomorphism $\psi : QH(M) \to HF(H, J)$ defined when the pair $(H, J))$ is generic. This suggests the definition of another product:

$$CF(M; H, J) \circledast \mathcal{C}(f, \rho, J) \to \mathcal{C}(f, \rho, J)$$

which, in homology, should be identified with (22) via the PSS map (here f, ρ, J are so that the respective pearl complex is well defined; $CF(M; H, J)$ is the periodic orbit Floer complex associated to (H, J) whose homology will be denoted by $HF(M; H, J)$). It is easy to see - as in [13] - that such a product can be defined by using pearls in which one of the disks is replaced by a Floer half tube parametrized by $(-\infty, 0] \times S^1$ with the $-\infty$ end on a periodic orbit of X_H and the $\{0\} \times S^1$ end on L.

If a Floer half tube u as above exists, then:

$$\int_{S^1 \times \{0\}} H(x, t) dt \leq \mathcal{A}_H(\bar{\gamma}) \ .$$

Here $lim_{s \to -\infty} u(s, -) = \gamma$, $\bar{\gamma}$ is the periodic orbit γ together with an appropriate capping and $\mathcal{A}_H$ is the Floer action functional.

The interest in this construction comes from noticing that, as a consequence of the remark above, if a class $a \in QH(M;\Lambda)$ acts non-trivially on $QH(L)$, then the spectral invariants associated to a can be bounded in terms of the behavior of the respective Hamiltonians on L. In turn, this has interesting geometric applications. We will not further discuss these issues here - the whole topic is described in detail in [13].

5. Applications I: topological rigidity of Lagrangian submanifolds

The purpose of this section is to show that, in a manifold with sufficiently rich quantum homology, even mild algebraic topological conditions imposed to monotone Lagrangians restrict considerably their homological or homotopical types.

5.1. Homological $\mathbb{R}P^n$'s.

Consider the complex projective space $\mathbb{C}P^n$ endowed with its standard Kähler symplectic structure ω_{FS}. Our first application deals with Lagrangians $L \subset \mathbb{C}P^n$ whose first integral homology $H_1(L;\mathbb{Z})$ satisfies $2H_1(L;\mathbb{Z}) = 0$, i.e. $\forall\, \alpha \in H_1(L;\mathbb{Z})$, $2\alpha = 0$. A familiar example of such a Lagrangian is $\mathbb{R}P^n \subset \mathbb{C}P^n$, $n \geq 2$. In fact, this is the only known example of a Lagrangian $L \subset \mathbb{C}P^n$ with this property. The following theorem shows that at least from the homological point of view this example is unique.

Theorem 5.1.1. *Let $L \subset \mathbb{C}P^n$ be a Lagrangian submanifold with $2H_1(L;\mathbb{Z}) = 0$.*

i. There exists a map $\phi : L \to \mathbb{R}P^n$ which induces an isomorphism of rings on $\mathbb{Z}_2$-homology: $\phi_* : H_*(L;\mathbb{Z}_2) \xrightarrow{\cong} H_*(\mathbb{R}P^n;\mathbb{Z}_2)$, the ring structures being defined by the intersection product. In particular we have $H_i(L;\mathbb{Z}_2) = \mathbb{Z}_2$ for every $0 \leq i \leq n$, and $H_*(L;\mathbb{Z}_2)$ is generated as a ring by $H_{n-1}(L;\mathbb{Z}_2)$.

ii. Denote by $h = [\mathbb{C}P^{n-1}] \in H_{2n-2}(\mathbb{C}P^n;\mathbb{Z}_2)$ the generator. Then $h \widetilde{\cap} [L]$ is the generator of $H_{n-2}(L;\mathbb{Z}_2)$. Here $\widetilde{\cap}$ stands for the exterior intersection product between elements of $H_*(\mathbb{C}P^n;\mathbb{Z}_2)$ and $H_*(L;\mathbb{Z}_2)$.

iii. Denote by $\mathrm{inc}_* : H_i(L;\mathbb{Z}_2) \to H_i(\mathbb{C}P^n;\mathbb{Z}_2)$ the homomorphism induced by the inclusion $L \subset \mathbb{C}P^n$. Then inc_* is an isomorphism for every $0 \leq i = \text{even} \leq n$.

In view of this theorem it is tempting to conjecture that the only Lagrangians $L \subset \mathbb{C}P^n$ with $2H_1(L;\mathbb{Z}) = 0$ are homeomorphic (or diffeomorphic) to $\mathbb{R}P^n$, or more daringly symplectically isotopic to the standard embedding of $\mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$.

Parts of Theorem 5.1.1 have been proved before by a variety of methods by Seidel [35] and later on by Biran [10]. We will now outline a different proof based on our theory. We refer the reader to [12, 13] for the full details of the proof.

We start with the following general observation. Recall that $QH(L)$ is a module over $QH(M;\Lambda)$. We have: *suppose that $a \in QH_q(M;\Lambda)$ is an invertible element (of pure degree q). Then the map $a \circledast (-)$ gives rise to* isomorphisms $QH_i(L) \longrightarrow QH_{i+q-2n}(L)$ *for every $i \in \mathbb{Z}$.* This clearly follows from the general algebraic notion of a "module over a ring with unit".

Outline of the proof of Theorem 5.1.1 (see [13] for the complete proof). A simple computation shows that L is monotone with $N_L = k(n+1)$ where k is either 1 or 2.

Denote by $h = [\mathbb{C}P^{n-1}] \in H_{2n-2}(\mathbb{C}P^n;\mathbb{Z}_2)$ the class of an hyperplane. Recall from Example 2.5.1 in §2.5 that $h \in QH_{2n-2}(\mathbb{C}P^n;\Lambda)$ is an invertible element. Therefore external multiplication by h gives isomorphisms: $h \circledast (-) : QH_i(L) \longrightarrow QH_{i-2}(L)$. In other words, $QH_*(L)$ is 2-periodic.

Choose a generic triple (f, ρ, J) with f having exactly one minimum x_0 and one maximum x_n. Denote by $(C(f,\rho), \partial_0)$ the Morse complex and by $\mathcal{C}(f,\rho,J) = C(f,\rho) \otimes \Lambda$ the pearl complex endowed with the pearly differential d. Recall from §2.7 that we can write $\partial = \partial_0 + \partial_1 t + \cdots + \partial_\nu t^\nu$, where each ∂_j is an operator that sends $C_*(f,\rho)$ to $C_{*-1+jN_L}(f,\rho)$.

We claim that $N_L = n+1$, i.e. $k = N_L/(n+1)$ must be 1. Indeed, if $k = 2$ then ∂_j, $j \geq 1$, must vanish since $-1 + jN_L = -1 + 2(n+1)j > n$. This implies that $d = \partial_0$, hence $QH_*(L) = (H(L;\mathbb{Z}_2) \otimes \Lambda)_*$. In particular, we have

$$QH_i(L) \cong H_i(L;\mathbb{Z}_2) \quad \forall\, 0 \leq i \leq n, \quad QH_j(L) = 0 \quad \forall\, n+1 \leq j \leq 2n+1.$$

However, this contradicts the 2-periodicity of $QH_*(L)$. This proves that $k = 1$.

As $N_L = n+1$, the differential d can be written as $d = \partial_0 + \partial_1 t$. For degree reasons $\partial_1(x) = 0$ for every critical point x with $|x| \geq 1$. As for $\partial_1(x_0)$, it can be either 0 or x_n. It follows that $QH_i(L) \cong H_i(L;\mathbb{Z}_2)$ for every $1 \leq i \leq n-1$. We claim that $\partial_1(x_0) = 0$ too. Indeed, if $\partial_1(x_0) = x_n$ then, as $\partial_0(x_0) = 0$, we have $d(x_0) = x_n t$ hence $[x_n] = 0 \in QH(L)$. But $[x_n]$ is the unity of $QH(L)$, so $QH_*(L) = 0$. In particular we have $H_1(L;\mathbb{Z}_2) \cong QH_1(L) = 0$. But this cannot happen since this would imply that $H_1(L;\mathbb{Z}) = 0$ (recall that $2H_1(L;\mathbb{Z}) = 0$) which would in turn imply that $N_L = 2C_{\mathbb{C}P^n} = 2(n+1)$, a contradiction. This proves that $\partial_1(x_0) = 0$. Therefore, we have $d \equiv \partial_0$, hence $QH(L) = H(L;\mathbb{Z}_2) \otimes \Lambda$.

Since $QH_*(L)$ is 2-periodic we obtain: $H_{2i}(L;\mathbb{Z}_2) \cong H_0(L;\mathbb{Z}_2) = \mathbb{Z}_2$, whenever $0 \leq 2i \leq n$. Similarly, $H_{2i+1}(L;\mathbb{Z}_2) \cong QH_{-1}(L) \cong QH_n(L) \cong H_n(L;\mathbb{Z}_2) = \mathbb{Z}_2$, whenever $1 \leq 2i+1 \leq n$. The isomorphism $QH_{-1} \cong QH_n$ here holds since QH_* is, by definition, also $N_L = n+1$ periodic. Summing up, we have $H_i(L;\mathbb{Z}_2) \cong \mathbb{Z}_2$ for every $0 \leq i \leq n$. Since $H_i(\mathbb{R}P^n;\mathbb{Z}_2) = \mathbb{Z}_2$ for every i, $0 \leq i \leq n$, this shows that $H_*(L;\mathbb{Z}_2) \cong H_*(\mathbb{R}P^n;\mathbb{Z}_2)$.

Notice that we actually have $QH_i(L) \cong \mathbb{Z}_2$ for each $i \in \mathbb{Z}$. Denote by $\alpha_i \in QH_i(L)$, $i \in \mathbb{Z}$, the corresponding generators. As $h \in QH_{2n-2}(\mathbb{C}P^n;\Lambda)$ is invertible we have $h \circledast \alpha_i = \alpha_{i-2}$, for every $i \in \mathbb{Z}$. For degree reasons it follows that $h \widetilde{\cap} \alpha_j = \alpha_{j-2}$ for every $2 \leq j \leq n$. (See the discussion in §2.7.) A similar argument shows that $\alpha_{n-2}^{\cap j} = \alpha_{n-2j}$ for every $0 \leq j \leq n/2$ and that $\alpha_{n-1} \cap \alpha_{n-2l} = \alpha_{n-2l-1}$ for every $0 \leq l \leq (n-1)/2$. Denote by $\alpha^i \in H^i(L;\mathbb{Z}_2)$ the generator. What we have just proved is equivalent to saying that α^2 generates $H^{\mathrm{even}}(L;\mathbb{Z}_2)$ (with respect to the cup product) and that $\alpha^1 \cup H^{\mathrm{even}}(L;\mathbb{Z}_2) = H^{\mathrm{odd}}(L;\mathbb{Z}_2)$.

By a purely topological argument (without any symplectic ingredients) one shows now that $\alpha^1 \cup \alpha^1 = \alpha^2$. (this can be proved e.g. by a Bockstein exact sequence using the fact that $H_1(L; \mathbb{Z})$ is a non-trivial 2-torsion group). Equivalently, this means that $\alpha_{n-1} \cap \alpha_{n-1} = \alpha_{n-2}$. Summing up the information up to now, we have that the $\mathbb{Z}_2$-homologies (resp. cohomologies) of L and $\mathbb{R}P^n$ are isomorphic as rings with respect to the cup (resp. intersection) products.

The statement on inc_* at point iii of the Theorem can be proved by similar arguments by looking at the quantum inclusion map $i_L : QH_*(L) \to QH_*(\mathbb{C}P^n; \Lambda)$.

Finally, the fact that the isomorphism $H_*(L; \mathbb{Z}_2) \cong H_*(\mathbb{R}P^n; \mathbb{Z}_2)$ is induced by a map $\phi : L \to \mathbb{R}P^n$ follows from general algebraic topology. The argument is as follows. Let $\bar{\phi} : L \to K(\mathbb{Z}_2, 1) = \mathbb{R}P^\infty$ be the classifying map associated to α^1, so that $\bar{\phi}^* c = \alpha^1$, where $c \in H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$ is a fundamental class. As $\dim L = n$, $\bar{\phi}$ factors through a map $\phi : L \to \mathbb{R}P^n$ which still satisfies $\phi^* \gamma^1 = \alpha^1$, where $\gamma^1 = c|_{\mathbb{R}P^n} \in H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ is the generator. As both α^1 and γ^1 generate their respective cohomology rings it immediately follows that $\phi^* : H^*(\mathbb{R}P^n; \mathbb{Z}_2) \to H^*(L; \mathbb{Z}_2)$ is an isomorphism. □

5.2. **Homological spheres in the quadric.** Another case which exemplifies topological rigidity is that of the quadric. Consider the smooth complex quadric $Q \subset \mathbb{C}P^{n+1}$ endowed with the induced symplectic structure from $\mathbb{C}P^{n+1}$. Note that Q contains Lagrangians with $H_1(L; \mathbb{Z}) = 0$, e.g. Lagrangian spheres. (To see this, write Q as $\{z_0^2 + \cdots + z_n^2 = z_{n+1}^2\}$ and take $L = Q \cap \mathbb{R}P^{n+1}$.) The next theorem shows that homologically this is the only example.

Theorem 5.2.1. *Let $L \subset Q$, $n \geq 2$, be a Lagrangian submanifold with $H_1(L; \mathbb{Z}) = 0$. Assume that $n = \dim_{\mathbb{C}} Q$ is even. Then $H_*(L; \mathbb{Z}_2) \cong H_*(S^n; \mathbb{Z}_2)$.*

The proof is based on similar ideas to that of Theorem 5.1.1. The main point now is that the class of a point $[pt] \in QH_0(Q; \Lambda)$ is invertible. See [12, 13] for a detailed proof. It is likely that a similar statement holds for n odd but our methods do not yield information in that case.

6. Applications II: existence of holomorphic disks and symplectic packing

In this section we explain how to use the theory of §2 in order to prove existence of holomorphic disks satisfying various incidence constrains. These in turn have applications to relative symplectic packing. Below we give a sample of our results in this direction. More complete and general results, as well as detailed proofs, can be found in [12, 13].

6.1. **Existence of disks with pointwise constrains.** Our first result deals with Lagrangians as in Theorem 5.1.1. We recall again the familiar example of $\mathbb{R}P^n \subset \mathbb{C}P^n$ for which we know for example that through every two points passes a real algebraic line, i.e. a holomorphic disk which is "half" of a projective line (hence has Maslov index $n+1$). The following theorem states, among other things, that this continues to be so for generic almost complex structures and moreover that it is actually true for all Lagrangians $L \subset \mathbb{C}P^n$ with 2-torsion $H_1(L;\mathbb{Z})$.

Theorem 6.1.1. *Let $L \subset \mathbb{C}P^n$ be a Lagrangian with $2H_1(L;\mathbb{Z}) = 0$. Then there exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}$ such that for every $J \in \mathcal{J}_{\text{reg}}$ the following holds:*

i. For every $p \in \mathbb{C}P^n \setminus L$ there exists a J-holomorphic disk $u : (D, \partial D) \to (\mathbb{C}P^n, L)$ with $u(\operatorname{Int} D) \ni p$ and $\mu([u]) = n + 1$.

ii. For every two distinct points $x, y \in L$ there exists a J-holomorphic disks u with $u(\partial D) \ni x, y$ and $\mu([u]) = n + 1$. The number of such disks u with $u(-1) = x$, $u(1) = y$, up to reparametrization is even.

iii. If $n = 2$ then for every $p \in \mathbb{C}P^2 \setminus L$ and $x, y \in L$ there exists a J-holomorphic disk u with $u(\operatorname{Int} D) \ni p$, $u(\partial D) \ni x, y$ and $\mu([u]) \leq 6$.

This theorem adds more evidence to the tempting conjecture, motivated by Theorem 5.1.1, that $L = \mathbb{R}P^n$ is in some sense the unique example of a Lagrangian in $\mathbb{C}P^n$ with 2-torsion first homology.

The next result is about the Clifford torus

$$\mathbb{T}_{\text{clif}} = \big\{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid |z_0| = \cdots = |z_n|\big\} \subset \mathbb{C}P^n.$$

This is a monotone Lagrangian torus with minimal Maslov number $N = 2$.

Theorem 6.1.2. *There exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}$ such that for every $J \in \mathcal{J}_{\text{reg}}$ the following holds:*

i. For every $p \in \mathbb{C}P^n \setminus \mathbb{T}_{\text{clif}}$ there exists a J-holomorphic disk u with $u(\operatorname{Int} D) \ni p$ and $\mu([u]) \leq 2n$.

ii. For every $x \in \mathbb{T}_{\text{clif}}$ there exists a J-holomorphic disk u with $u(\partial D) \ni x$ and $\mu([u]) = 2$.

iii. If $n = 2$ then for every $p \in \mathbb{C}P^2 \setminus \mathbb{T}_{\text{clif}}$ and $x \in \mathbb{T}_{\text{clif}}$ there exists a J-holomorphic disk u with $u(\operatorname{Int} D) \ni p$, $u(\partial D) \ni x$ and $\mu([u]) \leq 4$.

Finally, consider the smooth complex quadric $Q \subset \mathbb{C}P^{n+1}$ endowed with the induced symplectic structure from $\mathbb{C}P^{n+1}$.

Theorem 6.1.3. *Let $L \subset Q$ be a Lagrangian with $H_1(L;\mathbb{Z}) = 0$. Assume that $n = \dim L \geq 2$. Then there exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}$ such for every $J \in \mathcal{J}_{\text{reg}}$ the following holds:*

i. For every $p \in Q \setminus L$ and $x \in L$, there exists a J-holomorphic disk u with $u(\text{Int}\, D) \ni p$, $u(\partial D) \ni x$ and $\mu([u]) = 2n$.

ii. If $n =$ even then for every three distinct points $x, y, z \in L$ there exists a J-holomorphic disk u with $u(\partial D) \ni x, y, z$ and $\mu([u]) = 2n$.

We will outline the proofs of some of the theorems above in §6.3 below. Before doing this we present some immediate applications to symplectic packing.

6.2. **Relative symplectic packing.** Let (M^{2n}, ω) be a $2n$-dimensional symplectic manifold and $L \subset M$ a Lagrangian submanifold. Denote by $B(r) \subset \mathbb{R}^{2n}$ the closed $2n$-dimensional Euclidean ball of radius r endowed with the standard symplectic structure ω_{std} of $\mathbb{R}^{2n}$. Denote by $B_{\mathbb{R}}(r) \subset B(r)$ the "real" part of $B(r)$, i.e. $B_{\mathbb{R}}(r) = B(r) \cap (\mathbb{R}^n \times 0)$. Note that $B_{\mathbb{R}}(r)$ is Lagrangian in $B(r)$. By a *relative symplectic embedding* $\varphi : (B(r), B_{\mathbb{R}}(r)) \to (M, L)$ of a ball in (M, L) we mean a symplectic embedding $\varphi : B(r) \to (M, \omega)$ which satisfies $\varphi^{-1}(L) = B_{\mathbb{R}}(r)$. By analogy with the (absolute) Gromov width, we define here the Gromov width of $L \subset M$ to be

$$w(L) = \sup\{\pi r^2 \mid \exists \text{ a relative symplectic embedding } (B(r), B_{\mathbb{R}}(r)) \to (M, L)\}.$$

We will consider also symplectic embeddings of balls in the complement of L, i.e. symplectic embeddings $\psi : B(r) \to M \setminus L$. The Gromov width of $M \setminus L$ is:

$$w(M \setminus L) = \sup\{\pi r^2 \mid \exists \text{ a symplectic embedding } B(r) \to (M \setminus L)\}.$$

A natural generalization is to consider embeddings of several balls with pairwise disjoint images i.e. symplectic packing. Let $l, m \geq 0$ and $r_1, \ldots, r_l > 0$, $\rho_1, \ldots, \rho_m > 0$. A *mixed symplectic packing* of (M, L) by balls of radii $(r_1, \ldots, r_l; \rho_1, \ldots, \rho_m)$ is given by l relative symplectic embeddings $\varphi_i : (B(r_i), B_{\mathbb{R}}(r_i)) \to (M, L)$, $i = 1, \ldots, l$, and m symplectic embeddings $\varphi_j : B(r_j) \to M \setminus L$, $j = l+1, \ldots, l+m$, such that the images of all the φ_k's are mutually disjoint, i.e.: $\varphi_{k'}(B(r_{k'})) \cap \varphi_{k''}(B(r_{k''})) = \emptyset$ for every $k' \neq k''$.

The following proposition provides a link between symplectic packing and existence of holomorphic disks passing through given points. It is a straightforward generalization of Gromov's original approach to symplectic packing [25].

Proposition 6.2.1 (See [12, 13])**.** *Let $L \subset (M, \omega)$ be a Lagrangian submanifold and $E > 0$. Suppose that there exists a dense subset $\mathcal{J}_* \subset \mathcal{J}(M, \omega)$, a dense subset of m-tuples $\mathcal{U}' \subset (M \setminus L)^{\times m}$, and a dense subset of l-tuples $\mathcal{U}'' \subset L^{\times l}$ such that for every $J \in \mathcal{J}_*$, $(p_1, \ldots, p_m) \in \mathcal{U}'$, $(q_1, \ldots, q_l) \in \mathcal{U}''$ there exists a J-holomorphic disk $u : (D, \partial D) \to$*

(M, L) with $u(\operatorname{Int} D) \ni p_1, \ldots, p_m$, $u(\partial D) \ni q_1, \ldots, q_l$ and $\operatorname{Area}_\omega(u) \leq E$. Then for every mixed symplectic packing of (M, L) by balls of radii $(r_1, \ldots, r_l; \rho_1, \ldots, \rho_m)$ we have:

$$\sum_{i=1}^{l} \frac{\pi r_i^2}{2} + \sum_{j=1}^{m} \pi \rho_j^2 \leq E.$$

Combining Theorems 6.1.1– 6.1.3 with Proposition 6.2.1 we obtain the following packing inequalities. Below we normalize the symplectic structure ω_{FS} of $\mathbb{C}P^n$ so that $\int_{\mathbb{C}P^1} \omega_{\mathrm{FS}} = \pi$. With this normalization we have $(\mathbb{C}P^n \setminus \mathbb{C}P^{n-1}, \omega_{\mathrm{FS}}) \approx (\operatorname{Int} B^{2n}(1), \omega_{\mathrm{std}})$, hence $w(\mathbb{C}P^n) = 1$.

Corollary 6.2.2. *i. If $L \subset \mathbb{C}P^n$ is a Lagrangian with $2H_1(L; \mathbb{Z}) = 0$ then we have $w(\mathbb{C}P^n \setminus L) \leq \frac{1}{2}$.*

ii. For $\mathbb{T}_{\mathrm{clif}} \subset \mathbb{C}P^2$ we have $w(\mathbb{T}_{\mathrm{clif}}) = \frac{2}{n+1}$, $w(\mathbb{C}P^n \setminus \mathbb{T}_{\mathrm{clif}}) = \frac{n}{n+1}$.

iii. For every mixed symplectic packing of $(\mathbb{C}P^2, \mathbb{T}_{\mathrm{clif}})$ by two balls of radii $(r; \rho)$ we have $\pi r^2 + \frac{1}{2}\pi\rho^2 \leq \frac{2}{3}$.

iv. Let $L \subset Q$ be a Lagrangian with $H_1(L; \mathbb{Z}) = 0$, and assume that $n = \dim L =$ even. Then for every relative symplectic packing of (Q, L) by 3 balls of radii (ρ_1, ρ_2, ρ_3) we have $\pi(\rho_1^2 + \rho_2^2 + \rho_3^2) \leq 2$.

The phenomenon that the Gromov width may decrease after removing a Lagrangian submanifold was discovered in [7] where it was proved for example that $w(\mathbb{C}P^n \setminus \mathbb{R}P^n) = \frac{1}{2}$.

6.3. **How to prove existence of disks satisfying pointwise constrains.** We will outline here the proof of points i and ii of Theorem 6.1.1. We refer the reader to [13, 12] for the detailed proofs.

Let $L \subset \mathbb{C}P^n$ be a Lagrangian with $2H_1(L; \mathbb{Z}) = 0$. Recall from the proof of Theorem 5.1.1 that L is monotone with $N_L = n + 1$ and that $QH_i(L) \cong \mathbb{Z}_2$ for every $i \in \mathbb{Z}$. Denote by $\alpha_i \in QH_i(L)$ the generator. Note that $t \in \Lambda$ has $\deg t = -(n + 1)$ so that $QH_j(L) \cong QH_{j+n+1}(L)t$. In particular $\alpha_j = \alpha_{j+n+1}t$ for every $j \in \mathbb{Z}$.

Denote by $[pt] \in QH_0(\mathbb{C}P^n; \Lambda)$ the class of a point. Recall that $[pt]$ is an invertible element, hence we have $[pt] \circledast \alpha_i = \alpha_{i-2n}$ for every $i \in \mathbb{Z}$. In particular

$$[pt] \circledast \alpha_n = \alpha_{-n} = \alpha_1 t. \tag{23}$$

Let $f : L \to \mathbb{R}$ be a Morse function with one maximum x_n and $h : \mathbb{C}P^n \to \mathbb{R}$ a Morse function with one minimum at the point p. Choose two Riemannian metrics ρ_L and ρ_M. We make these choices so that (h, ρ_M, f, ρ_L) satisfy the Assumption 3.1.1 in §3.1. Choose a generic $J \in \mathcal{J}$. With these choices we have $[pt] = [p]$, $\alpha_n = [x_n]$. From (23) it follows that there exists a critical point $x_1 \in \operatorname{Crit}(f)$ of index $|x_1| = 1$ and $A \in H_2^D(M, L)$ with $\mu(A) = n+1$ such that the moduli space $\mathcal{P}_{\mathrm{prl}}(p, x_n, x_1; A; h, \rho_M, f, \rho_L, J)$, introduced

in §2.5, is non-empty. Let $(u_1, \ldots, u_l; k) \in \mathcal{P}_{\text{prl}}(p, x_n, x_1; A; h, \rho_M, f, \rho_L, J)$. By definition the J-holomorphic disk u_k satisfies $u_k(0) \in W^u_p(h)$. Since p is the minimum of h, we have $W^u_p(h) = \{p\}$, hence $u_k(0) = p$. Clearly $\mu([u_k]) \leq \mu(A) = n+1$. But as u_k can not be constant we actually have $\mu([u_k]) = n+1$. The disk $u = u_k$ satisfies the statement in point i of Theorem 6.1.1.

We turn to the proof of the statement at point ii of the theorem. Recall from the proof of Theorem 5.1.1 that $QH_*(L) \cong (H(L; \mathbb{Z}_2) \otimes \Lambda)_*$. Recall also that $\alpha_{n-1} \cap \alpha_{n-1} = \alpha_{n-2}$, where $\cap$ is the classical intersection product on $H_*(L; \mathbb{Z}_2)$.

We now claim that $\alpha_0 \circ \alpha_0 = \alpha_1 t$. To see this, first note that for degree reasons we have $\alpha_{n-1} \circ \alpha_{n-1} = \alpha_{n-1} \cap \alpha_{n-1} = \alpha_{n-2}$. Therefore:

$$\begin{aligned} \alpha_0 \circ \alpha_0 &= ([pt] \circledast \alpha_{n-1} t^{-1}) \circ ([pt] \circledast \alpha_{n-1} t^{-1}) = [pt] \circledast ([pt] \circledast (\alpha_{n-1} \circ \alpha_{n-1})) t^{-2} \\ &= [pt] \circledast ([pt] \circledast \alpha_{n-2}) t^{-2} = [pt] \circledast \alpha_{-1} t^{-1} = \alpha_1 t. \end{aligned}$$

Pick a generic triple of Morse functions f, f', f'' on L such that f and f' each have a single minimum, f at x and f' at y. Then, we have that $[x] \in QH_0(L; f, \rho_L, J)$ and $[y] \in QH_0(L; f', \rho_L, J)$ both represent $\alpha_0 \in QH_0(L)$. As $\alpha_0 \circ \alpha_0 = \alpha_1 t$, it follows that there exists $z \in \text{Crit}(f'')$ with $|z| = 1$ and $A \in H_2^D(M, L)$ with $\mu(A) = n+1$ such that the moduli space $\mathcal{P}_{\text{prod}}(x, y, z; A; f, f', f'', \rho_L, J)$, introduced in §2.4, is non-empty. As x and y are both minima of their functions it easily follows that for every element $(\mathbf{u}, \mathbf{u}', \mathbf{u}'', v) \in \mathcal{P}_{\text{prod}}(x, y, z; A; f, f', f'', \rho_L, J)$ we have $\mathbf{u}, \mathbf{u}', \mathbf{u}'' \equiv \text{const}$ hence the J-holomorphic disk v satisfies $x, y \in v(\partial D)$ and $\mu([v]) = n+1$. The point is again that as x is a minimum we have $W^u_x = \{x\}$ and similarly for y. The disk v satisfies the properties stated at point ii of the theorem (where the claimed disk was called u).

It remains to show that the number of such disks it even. To prove this pick a Morse function $g : L \to \mathbb{R}$ with a single minimum at x, and a single maximum at y. Write the pearl differential d as $d = \partial_0 + \partial_1 t$ as in the proof of Theorem 5.1.1. Clearly $\partial_1(x)$ counts the number of J-holomorphic disks (up to reparametrization) $u : (D, \partial D) \to (\mathbb{C}P^n, L)$ with $u(-1) = x$ and $u(1) = y$. However, as we saw in the proof of Theorem 5.1.1 we have $\partial_1 = 0$, hence the number of these disks is even. □

A proof of the statement at point iii of Theorem 6.1.1 as well as proofs of Theorems 6.1.2 and 6.1.3 can be found in [12, 13].

7. Applications III: relative enumerative invariants for Lagrangian tori

The purpose of this section is to present a general scheme that can be used to construct numerical invariants associated to monotone Lagrangians which is based on our machinery. We apply this scheme to the case of 2-dimensional tori.

7.1. **How to produce relative numerical invariants for wide Lagrangians.** We will assume that L is a monotone Lagrangian such that $QH(L) \cong H_*(L;\mathbb{Z}_2) \otimes \Lambda$. Such Lagrangians are called *wide* and it has been shown in [12] (see also [13]) that a large class of Lagrangians, tori in particular, can only be *narrow* - in the sense that $QH(L) = 0$ - or wide.

With this assumption, and supposing that the quantum product in $QH(L)$ is known, the naive way to produce numerical invariants would be to replicate the construction in the closed case: pick first a basis $\{a_i\}$ for $H_*(L; Z_2)$ and express the quantum product as $a_i \circ a_j = \sum s(i,j;h:k) a_h t^k$ with $s(i,j;h:k) \in \mathbb{Z}_2$; secondly, interpret $s(i,j;h:k)$ as the (algebraic) number of J-holomorphic disks of Maslov class kN_L through any cycles representing the classes a_i, a_j, a_h^* (where a_h^* is the dual of a_h). This strategy fails for two reasons and it is instructive to understand them in detail.

The first reason is quite obvious: the pearl moduli spaces consist of configurations involving not only a single J-holomorphic curve but also chains of curves joined together by Morse flow lines. As a consequence, the structural constants $s(i,j;h:k)$ can not be interpreted directly as counts of disks with pointwise constraints. Recall also that the reason these configurations of chains of curves are needed is that moduli spaces of disks have co-dimension one boundaries.

The second reason is much less obvious: the identification between $QH(L)$ and $H_*(L;\mathbb{Z}_2) \otimes \Lambda$ is not canonical and so the constants $s(i,j;h:k)$ as defined above are, in fact, not invariant ! This is a more subtle phenomenon and to describe it more precisely we will now assume additionally that L admits a perfect Morse function $f : L \to \mathbb{R}$. In this case, the isomorphism $QH(L) \cong H_*(L;\mathbb{Z}_2) \otimes \Lambda$ translates to the fact that the pearl complex $\mathcal{C}(f,\rho,J)$ (when defined) has a vanishing differential. In particular, each critical point $x \in \mathrm{Crit(f)}$ represents not only a singular homology class (because f is perfect) but also a quantum homology class. Assume now that f' is another perfect Morse function so that the pearl complex $\mathcal{C}(f',\rho,J)$ is also defined (obviously, it also has a vanishing differential). We already know from §3.4 that there is a chain morphism $\psi_{f',f} : \mathcal{C}(f,\rho,J) \to \mathcal{C}(f',\rho,J)$ which induces a canonical isomorphism in homology. In our case, as the pearl differentials vanish, $\psi_{f',f}$ is itself a canonical isomorphism. In general, this isomorphism has the form $\psi_{f',f} = \psi^M_{f',f} + t\psi^Q_{f',f}$ where $\psi^M_{f',f}$ is the Morse comparison morphism. Now, the key point here - and this is specific to the "open" case - is that the quantum contribution $\psi^Q_{f',f}$ is in general not zero ! Thus, while the structural constants of the quantum product $QH(L) \otimes QH(L) \to QH(L)$ are obviously invariant they can not be seen directly as invariants counting pearly configurations through singular cycles because, even if two

singular cycles represent the same singular homology class, they might represent different quantum classes.

We now describe a strategy which bypasses the two difficulties described above and leads to numerical invariants. We emphasize that, for the moment, this is a strategy and not an algorithm.

We will continue to assume that L is wide and admits a perfect Morse function if this last property is not satisfied, there is a purely algebraic minimal model technique which can be used instead [13][12].

Our approach consists of two steps which we describe below. Both depend only on the minimal Maslov number $N = N_L$ and of the singular homology $\mathcal{H} = H_*(L;\mathbb{Z}_2)$ of L. We fix some algebraic notation. We put $\Lambda = \mathbb{Z}_2[t, t^{-1}]$ and let $\Lambda^+ = \mathbb{Z}_2[t]$ with $deg(t) = -N$. For a free Λ^+-module V, let $Aut_0^+(V)$ be the Λ^+-automorphisms ξ of V, $\xi : V \to V$, which preserve degree and verify $\xi|_{V/tV} = id$. Let now $V' = V \otimes_{\Lambda^+} \Lambda$. This is clearly a free Λ-module. We denote by $Aut_0(V')$ the Λ-module automorphisms of V' which are in the image of $Aut_0^+(V)$.

i. Pick a basis $\{a_i\}$ for $\mathcal{H}\otimes\Lambda$ (as Λ module) and write the general form of the quantum product $(\mathcal{H} \otimes \Lambda) \otimes (\mathcal{H} \otimes \Lambda) \to (\mathcal{H} \otimes \Lambda)$ in this basis. The structural constants $s(i, j; h : k)$ appear as discussed above. Find expressions $E(\ldots, s(i, j; h : k), \ldots)$ written in the constants $s(i, j; h : k)$ which are invariant by all the automorphisms $\xi \in Aut_0(\mathcal{H} \otimes \Lambda)$ in the sense that, if the structural constants in the basis $\xi(a_i)$ are $s'(i, j; h : k)$, then $E(\ldots, s(i, j; h : k), \ldots) = E(\ldots, s'(i, j; h : k), \ldots)$ (even if not all $s'(i, j; h : k) = s(i, j; h : k)$).

ii. Let $f, f', f'' : L \to \mathbb{R}$ be perfect Morse functions and let ρ, J be generic. Use the pearly description of the product

$$\circ : \mathcal{C}(f, \rho, J) \otimes \mathcal{C}(f', \rho, J) \to \mathcal{C}(f'', \rho, J)$$

to provide a geometric interpretation of the invariant expressions detected at point i. in terms of counts of disks with various incidence conditions (in general, disks of different Maslov classes will appear in the same count).

As the counts given at point ii. are left invariant by the automorphisms of $QH(L)$ which are induced by the comparisons which appear at changes of the data (f, ρ, J), it follows that each of them provides a numerical invariant for all Lagrangians of Maslov class N and singular homology $\mathcal{H}$ (i.e. a number independent of f, f', f'', J, ρ). It is important to emphasize that due to the associativity of the quantum product the structural constants

$s(i, j; h : k)$ are not independent and this is a source of relations among the various invariants constructed as above.

Once the two steps above are achieved, computing the invariants for a specific Lagrangian with the fixed homology $\mathcal{H}$ and Maslov class N reduces to the computation of the quantum product $QH(L) \otimes QH(L) \to QH(L)$.

This approach will be pursued systematically elsewhere. We will describe it here only in the case of 2-tori.

7.2. **Numerical invariants for tori.** It turns out (see again [12],[13]) that for a monotone Lagrangian torus T to be wide, the Maslov number N_T has to be equal to 2 so we assume this here. Thus, to implement the step i. in our strategy we fix a basis m, a, b, w for $H_*(T; \mathbb{Z}_2) \otimes \Lambda$ so that a, b form a basis for $H_1(T; Z_2)$, $w \in H_2(T; \mathbb{Z}_2)$ is the generator and $m \in (H_*(T; \mathbb{Z}_2) \otimes \Lambda)_0$ together with wt form a basis for $H_0(T; \mathbb{Z}_2) \oplus H_2(T; \mathbb{Z}_2)t$. Notice that, in this case, for degree reasons, the isomorphism $QH_1(L) \cong H_1(L; \mathbb{Z}_2) \otimes \Lambda$ is canonical.

We now fix the notation for the structural constants involved in the quantum product. To do so we recall that this product is a deformation of the usual intersection product at the chain level (hence, in this case also at the homology level) and that w is the unit.

We now write: $a \circ a = \alpha wt$, $b \circ b = \beta wt$, $a \circ b = m + \gamma' wt$, $b \circ a = m + \gamma'' wt$ and we use the associativity of the quantum product to deduce:

$$\begin{aligned} & m \circ a = \alpha bt + \gamma'' at, \quad a \circ m = \alpha bt + \gamma' at \\ & m \circ b = \beta at + \gamma' bt, \quad b \circ m = \beta at + \gamma'' bt \\ & m \circ m = (\gamma' + \gamma'')mt + (\alpha\beta + \gamma'\gamma'')wt^2. \end{aligned} \tag{24}$$

For further use, we fix the notation $s_1 = \gamma' + \gamma''$ and $s_2 = \alpha\beta + \gamma'\gamma''$. Let $\xi \in Aut_0(H_*(T; \mathbb{Z}_2) \otimes \Lambda)$. There are only two possibilities for such an automorphism as, for degree reasons, the only quantum contribution in ξ can appear in $\xi(m) = m + \epsilon wt$, $\epsilon \in \mathbb{Z}_2$. Let ξ_1 be the automorphism for which $\epsilon = 1$ (when $\epsilon = 0$ the corresponding automorphism is the identity). It is immediate to see that, for degree reasons α and β are invariant with respect to ξ_1 and thus, they are invariant in the sense of the step i. of §7.1. Let us remark that $\gamma' + \gamma''$ is also invariant in the same sense. To see this write

$$\xi_1(m) \circ \xi_1(m) = m \circ m + wt^2 = s_1 mt + (s_2 + 1)wt^2 = s_1 \xi_1(m)t + (s_1 + s_2 + 1)\xi_1(w)t^2 \tag{25}$$

and so the structural constant $s_1 = \gamma' + \gamma''$ is invariant. At the same time, individually, the constants γ', γ'' are not necessarily invariant: indeed, if $\gamma' = 1$, we have $\xi_1(a) \circ \xi_1(b) = \xi_1(m)$ (while, for invariance, we would need $\xi_1(a) \circ \xi_1(b) = \xi_1(m) + \gamma' \xi_1(w)t$).

We now can proceed to the second step described in §7.1 and provide a geometric description for each of these three invariants α, β and $\gamma' + \gamma''$.

Fix a basis a', b' of the *integral* homology $H_1(T;\mathbb{Z})$ which correspond after mod 2 reduction to the a, b above. Fix a point $x \in T$ and for some almost complex structure J compatible with ω let $\mathcal{E}_2(x)$ be the set of J-holomorphic disks u with boundary on T passing through x and with $\mu([u]) = 2$. Define a function $\nu : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_2$ as follows:

$$\nu(k,l) = \#_{\mathbb{Z}_2}\{u \in \mathcal{E}_2(x) \mid [u(\partial D)] = ka' + lb'\}. \tag{26}$$

where $J \in \mathcal{J}_{\text{reg}}$ is a generic almost complex structure. As 2 is the minimal Maslov class, $\nu(k,l)$ does not depend on the choice of $J \in \mathcal{J}_{\text{reg}}$ or on the choice of the point x (this follows by a standard cobordism argument). Moreover, $\nu(k,l) = 0$ for all but a finite number of pairs (k,l).

Theorem 7.2.1. *(see [12]) The coefficients α, β are given by:*

$$\alpha = \sum_{k,l} \nu(k,l) \frac{l(l+1)}{2} \, (\text{mod} 2), \quad \beta = \sum_{k,l} \nu(k,l) \frac{k(k+1)}{2} \, (\text{mod} 2).$$

The sum $\gamma' + \gamma''$ is given by:

$$\gamma' + \gamma'' = \sum_{k,l} \nu(k,l) kl \, (\text{mod} 2).$$

Notice also that $\gamma' + \gamma''$ is precisely the obstruction to the commutativity of the quantum product. Moreover, when this product is non-commutative (thus when $\gamma' + \gamma'' = 1$) we have from formula (25) that s_2 si also an invariant and is equal to $\alpha\beta$.

Till now the geometric interpretation of both s_1 and s_2 has been based only on the formulae (24) which, in turn, are based on the associativity of the quantum product. However, - as indicated at the step ii. in §7.1 - both s_1 and s_2 have also geometric interpretations based directly on the definition of the quantum product $m \circ m$. We describe these interpretations next.

Let Δ be a triangle embedded in the torus T with vertices A, B, C and with edges AB, BC, CA. For a fixed, generic almost complex structure J let n_Δ be the number (mod 2) of disks of Maslov class 4 passing, *in order* through the three points A, B, C. Let n_A be the number mod 2 (up to reparametrization) of J-disks u of Maslov class class 2 with boundary on L and with $u(-1) = A$, $u(+1) \in BC$ (for generic J both numbers are finite and the intersections of the disks going through A with the opposite edge is transverse). Similarly, let n_B, n_C be the same numbers associated to the other vertices of Δ.

Theorem 7.2.2. *(see [12]) We have the formulae:*

$$s_1 = n_A + n_B + n_C$$

Thus, the sum $n_A + n_B + n_C$ is independent of J and Δ. If $s_1 = 1$, then s_2 is invariant and it equals

$$s_2 = n_\Delta + n_B n_C \ .$$

Thus, in this case, $n_\Delta + n_B n_C$ is also independent of J and Δ and equals the product $\alpha\beta$.

Remark 7.2.3. a. An interesting consequence of the formulae above is that if the quantum multiplication in $QH(L)$ is non-commutative, then the number of J-holomorphic disks of Maslov index 4 passing in order through any three distinct points A, B, C in L can be computed out of the numbers n_A, n_B, α, β which only involve Maslov 2 disks. Moreover, the term $n_B n_C$ is exactly the correction needed to be added to the number of Maslov 4 disks to obtain an invariant.

b. Another nice consequence is that, for the same type of monotone Lagrangian torus as at point a. (i.e. $s_1 = 1$) the number of disks of Maslov class 4 through any three points is always even. Indeed, for a triangle $\Delta = ABC$ as above let n'_Δ be the number of such disks going *in order* through A, C, B. Clearly, we have $s_2 = n'_\Delta + n_C n_B$. Thus the total number (mod 2) of disks of Maslov 4 through the three points is $n_\Delta + n'_\Delta = 2s_2 + 2n_C n_B = 0 \in \mathbb{Z}_2$.

Sketch of proof of $s_2 = n_\Delta + n_B n_C$. We refer to [12] for the rest of the proof of the theorem and for additional details. Let $f, g : T \to \mathbb{R}$ be two perfect Morse functions with pairwise distinct critical points. Let x_0 be the minimum of f, let x_2 be the maximum of f, let y_0 be the minimum of g. We may assume that the choices of f, g as well as that of the Riemannian metric ρ are such that $y_0 = A$, $x_0 = B$, $x_2 = C$ and the edge CA is the the unique flow line of $-\nabla f$ going from x_2 to y_0, and (after slightly rounding the corner at A) the edge AB is the unique flow line going from y_0 to x_0.

Notice that the product $\circ$ defined in §2.4 is also defined when $f'' = f$. In our case, the product we are interested in is:

$$\mathcal{C}(f, \rho, J) \otimes \mathcal{C}(g, \rho, J) \to \mathcal{C}(f, \rho, J)$$

and we want to list all the configurations which give the coefficient of $x_2 t^2$ in $x_0 \circ y_0$. It is not hard to see that there are precisely two types of such configurations:

i. disks of Maslov class 4 passing, in order through x_0, x_2, y_0.
ii. configurations made out of a disk of Maslov class 2 going through x_0 followed by a negative gradient flow line of f going through y_0 which continues till it reaches a second disk of Maslov class 2 which goes through x_2.

Clearly, the number of configurations of type i. is precisely n_Δ. A little thought (and a look at Figure 6) shows that the configurations of type ii. are precisely those counted by $n_B n_C$.

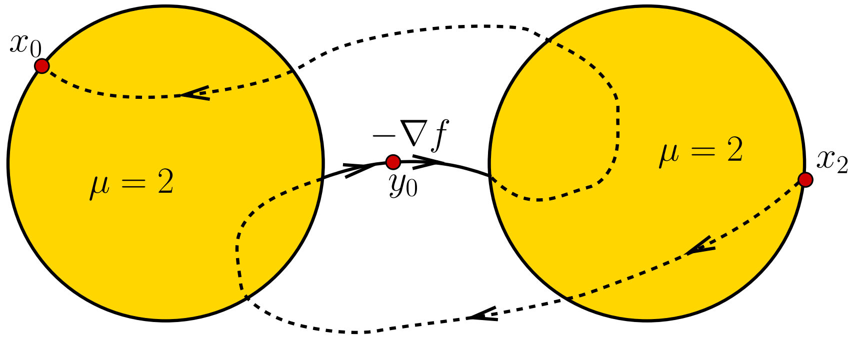

FIGURE 6.

Finally, we can take as generators of $H_*(T;\mathbb{Z}_2)\otimes\Lambda$ the critical points of f. In this case, we know that s_2 is the coefficient of x_2t^2 in the product $x_0 \circ x_0$ (written now in quantum homology and not at the chain level). In homology the relation between x_0 and y_0 is $y_0 = x_0 + \epsilon' x_2 t$ where $\epsilon' \in \mathbb{Z}_2$. Therefore, $x_0 \circ y_0 = x_0 \circ x_0 + \epsilon' x_0 t$ so that s_2 coincides with the coefficient of x_2t^2 in $x_0 \circ y_0$ which is $n_\Delta + n_B n_C$. $\square$

Example 7.2.4. We will give here a couple of examples for the invariants discussed in this section (see [12] for details on these calculations).

a. The Clifford torus, $\mathbb{T}^2_{\text{clif}} = \{[z_0 : z_1 : z_2] \in \mathbb{C}P^2 : |z_0| = |z_1| = |z_2|\}$. In this case we have $\alpha = \beta = \gamma' + \gamma'' = 1$. Therefore, $s_1 = 1$ and $s_2 = 1$.

b. The split torus in $S^2 \times S^2$. This is the split torus $Eq \times Eq \subset (S^2 \times S^2, \omega_{S^2} \times \omega_{S^2})$ where Eq is the equator in S^2. In this example, $\alpha = \beta = 1$, $\gamma' + \gamma'' = 0$. Thus in this case s_2 is not necessarily invariant.

7.2.1. *Relation to previous works.* An explicit computation of the Floer homology of the Clifford torus was first carried out by Cho [15]. Computations related to the quantum product for the Clifford torus have been done before by Cho [16] and by Cho and Oh [17] using different methods (see also the recent work of Fukaya, Oh, Ohta and Ono [22]). These works consider Lagrangian tori that appear as fibres of the moment map in a toric manifold, and the toric picture plays there a crucial role. It seems likely that these computations combined with our approach can give rise to more relative numerical invariants. It would be interesting to see if this leads to a better understanding of the structure and nature of these relative invariants.

8. Applications IV: from quantum structures to Lagrangian intersections

Here we explore the relations between the quantum operations from §2 associated to two different Lagrangians L and L'. It turns out that a correct composition of the operations involving $QH(L)$, $QH(L')$ and $QH(M)$ yields information on intersection properties of L and L'. The exposition presented here is somewhat heuristic in the sense that we ignore quite a few non-trivial technical difficulties and concentrate only on the geometric and algebraic pictures. For this reason, some of results below are marked with a * to indicate that their proofs are still not 100% rigorous. A rigorous treatment of the material of this section will be pursued in [11]. See also [13] for a different approach which is completely rigorous.

8.1. **Detecting Lagrangian intersections.** Let $L, L' \subset (M,\omega)$ be two monotone Lagrangians with minimum Maslov numbers N_L and $N_{L'}$. Denote by $\widetilde{\Lambda}^+_L$ and $\widetilde{\Lambda}^+_{L'}$ the corresponding rings as defined in §4.2 and let $\Lambda_L = \mathbb{Z}_2[t_0^{-1}, t_0]$, $\Lambda_{L'} = \mathbb{Z}_2[t_1^{-1}, t_1]$ be the associated Laurent polynomial rings so that $\deg t_0 = -N_L$ and $\deg t_1 = -N_{L'}$. Denote by $\Lambda_{L,L'}$ the ring $\Lambda_L \otimes_\Gamma \Lambda_{L'}$, where $\Gamma = \mathbb{Z}_2[s^{-1}, s]$ and Λ_L, Λ_L are Γ-modules by the maps $s \to t_0^{2C_M/N_L} \in \Lambda_L$ and $s \to t_1^{2C_M/N_{L'}} \in \Lambda_{L'}$. The ring $\Lambda_{L,L'}$ has a grading induced by both factors and it is easy to see that it is well defined. Note that

$$\Lambda_{L,L'} \cong \mathbb{Z}_2[t_0^{-1}, t_1^{-1}, t_0, t_1]/\{t_0^{2C_M/N_L} = t_1^{2C_M/N_{L'}}\}. \tag{27}$$

The map $q : \widetilde{\Lambda}^+_L \to \Lambda_{L,L'}$ defined by $q(T^A) = t_0^{\mu(A)/N_L}$ turn $\Lambda_{L,L'}$ into a commutative $\widetilde{\Lambda}^+_L$-algebra and similarly $\Lambda_{L,L'}$ is also a commutative $\widetilde{\Lambda}^+_{L'}$-algebra. According to the discussion in §4.2 we can define $QH(L;\Lambda_{L,L'})$ and $QH(L';\Lambda_{L,L'})$ as well as $QH(M;\Lambda_{L,L'})$ and all the theory from §2 continues to work in this setting. Note that the identifications Θ of $QH(L;\Lambda_{L,L'})$ and $QH(L';\Lambda_{L,L'})$ with $HF(L,L;\Lambda_{L,L'})$ and $HF(L',L';\Lambda_{L,L'})$ hold too since $\Lambda_{L,L'}$ is also a commutative $\mathbb{Z}_2[H_2^D(M,L)]$-algebra as well as a commutative $\mathbb{Z}_2[H_2^D(M,L')]$-algebra, both structures being compatible with the $\widetilde{\Lambda}^+_L$ and $\widetilde{\Lambda}^+_{L'}$-algebras structures. See point ii of Remark 4.2.2 in §4.2.

Let $i_L : QH_*(L;\Lambda_{L,L'}) \longrightarrow QH_*(M;\Lambda_{L,L'})$ be the quantum inclusion map (see §2.6) and let $j_{L'} : QH_*(M;\Lambda_{L,L'}) \longrightarrow QH_{*-n}(L';\Lambda_{L,L'})$ the map defined by $j_{L'}(a) = a \circledast [L']$.

The following theorem gives information on the composition

$$j_{L'} \circ i_L : QH_*(L;\Lambda_{L,L'}) \longrightarrow QH_{*-n}(L';\Lambda_{L,L'}) \ .$$

We denote by $\mathrm{Symp}_H(M,\omega)$ the group of symplectic diffeomorphisms of (M,ω) that act as the identity on $H_*(M)$. Note that we have $\mathrm{Symp}_H \supset \mathrm{Symp}_0 \supset \mathrm{Ham}$ where Symp_0 is the identity component of the symplectomorphism group.

Theorem 8.1.1. *Suppose that there exists* $\varphi \in \mathrm{Symp}_H(M, \omega)$ *such that* $L \cap \varphi(L') = \emptyset$. *Then* $j_{L'} \circ i_L = 0$.

A proof of this Theorem appears in [13], based on the relation between quantum structures and spectral invariants. In §8.2 below we will explain a completely different way to prove this theorem which yields more information on Lagrangian intersections. Before that, let us present two quick applications to Lagrangian intersections.

Corollary 8.1.2. *Let* $L, L' \subset \mathbb{C}P^n$ *be two monotone Lagrangians. If* $QH(L) \neq 0$ *and* $QH(L') \neq 0$, *then* $L \cap L' \neq \emptyset$.

This corollary has recently been obtained by Entov and Polterovich [20, 19], as well as by the authors of this paper in [13] by completely different methods based on the the tools developed in §2 and the theory of spectral numbers for Hamiltonian diffeomorphisms along the lines mentioned in §4.4, see also [3] for earlier results in this direction.

Proof of Corollary 8.1.2. As $QH(L), QH(L') \neq 0$ it is easy to see that we also have $QH(L; \Lambda_{L,L'}), QH(L'; \Lambda_{L,L'}) \neq 0$.

Let $f : L \longrightarrow \mathbb{R}$ be a Morse function with one minimum x_0. Let ρ be a Riemannian metric on L and $J \in \mathcal{J}$ an almost complex structure. Denote by d^L the pearl differential of the complex $\mathcal{C}(L; f, \rho, J)$.

Notice that although $x_0 \in \mathcal{C}_0(f, \rho, J)$ is a "Morse homology"-cycle it might not be a d^L-cycle. However, an argument based on duality (see §4.1.2) and the fact that $QH(L; \Lambda_{L,L'}) \neq 0$ implies that there exist $x_j \in \mathrm{Crit}_{jN_L}(f)$, $r_j \in \mathbb{Z}_2$, for $j \geq 1$ such that

$$x_0 + \sum_{j \geq 1} r_j x_j t_0^j \in \mathcal{C}_0(L; f, \rho, J),$$

is a d^L-cycle. (See [13] for more details.)

Denote by α_0 the homology class of this element. Consider the image of α_0 by the canonical map $QH(L) \to QH(L; \Lambda_{L,L'})$ induced by the obvious map $\Lambda_L \to \Lambda_{L,L'}$. We continue to denote this class by α_0. We will prove below that $j_{L'} \circ i_L(\alpha_0) \neq 0$.

First notice that

$$i_L(\alpha_0) = [pt] + \sum_{i \geq 1} a_i t_0^i, \tag{28}$$

where $a_i \in H_{iN_L}(M; \mathbb{Z}_2)$ and the sum is taken over all $0 < i$ with $iN_L \leq 2n$. The reason for the term $[pt]$ comes from the fact the the quantum inclusion extends (on the chain level) the classical map induced by the inclusion $L \to M$. The fact that there are no t_1's on the righthand side of (28) is because α_0 is the image of an element in $QH(L) = QH(L; \Lambda_L)$.

Applying the map $j_{L'}$ to (28) we obtain:

$$j_{L'} \circ i_L(\alpha_0) = [pt] \circledast [L'] + \sum_{i \geq 1} a_i \circledast [L'] t_0^i. \tag{29}$$

Now assume by contradiction that $j_{L'} \circ i_L(\alpha_0) = 0$. Since $[pt] \in QH_0(M; \Lambda_{L,L'})$ is invertible and $[L'] \neq 0$ (as $QH(L') \neq 0$) it follows that $[pt] \circledast [L'] \neq 0 \in QH_{-n}(L'; \Lambda_{L,L'})$. Next note that the products $[pt] \circledast [L']$ and $a_i \circledast [L']$ on the righthand side of (29) both belong to the image of $QH(L'; \Lambda_{L'}) \to QH(L'; \Lambda_{L,L'})$. As the sum on the righthand side of (29) vanishes it follows that there exists an index i that contributes to this sum such that $t_0^i = t_1^r$ for some $r \geq 1$. This can happen only if $\frac{2C_{\mathbb{C}P^n}}{N_L} = \frac{2n+2}{N_L}$ divides i. This implies that $2n + 2 \mid iN_L$, in particular $iN_L \geq 2n + 2$. On the other hand the i's that contribute to the sum in (28) (hence in (29)) all satisfy $iN_L \leq 2n$, a contradiction. This proves that $j_{L'} \circ i_L \neq 0$. The fact that $L \cap L' \neq \emptyset$ follows now from Theorem 8.1.1. □

Corollary* 8.1.3. *Let $L, L' \subset Q$ be two Lagrangians in the quadric with $H_1(L; \mathbb{Z}) = 0$, $H_1(L'; \mathbb{Z}) = 0$ and assume that both L and L' are relative spin (see [23] for the definition); e.g. both L and L' are Lagrangian spheres. Then $L \cap L' \neq \emptyset$.*

The statement of this corollary has been conjectured by Biran in [9, 8, 10].

Proof of Corollary 8.1.3. The proof below uses $\mathbb{Z}$ as the ground ring of coefficients. As already mentioned in §4.2.2 we expect our theory to work over $\mathbb{Z}$ however we have not rigorously checked that. Still, it is instructive to see how the proof works in this framework. Note that under the assumptions of the corollary, both L and L' are orientable and relative spin.

Put $2n = \dim Q$. As the minimal Chern number C_Q of Q is n we have $N = N_L = N_{L'} = 2n$. It follows that the ring $\Lambda_{L,L'} = \Lambda_L \otimes_\Gamma \Lambda_{L'}$ coincides with both of Λ_L and $\Lambda_{L'}$, i.e. it is $\mathbb{Z}_2[t^{-1}, t]$, where $\deg t = -2n$. We therefore denote all these rings by Λ and omit it from the notation.

As $N = 2n > n + 1$ there exists a canonical isomorphism $QH_*(L) \cong (H(L; \mathbb{Z}_2) \otimes \Lambda)_*$ and similarly for L'. Denote by $\alpha_0 \in H_0(L; \mathbb{Z})$ and by $[pt] \in H_0(Q; \mathbb{Z}_2)$ the classes of a point. By the results of [12, 13] we have $i_L(\alpha_0) = [pt] - [Q]t$, and $[pt] \circledast [L'] = -[L']t$. It follows that :

$$j_{L'} \circ i_L(\alpha_0) = ([pt] - [Q]t) \circledast [L'] = -2[L']t \neq 0.$$

The result now follows from Theorem 8.1.1. □

8.2. **A chain homotopy.** Let $f : L \longrightarrow \mathbb{R}$, $f' : L' \longrightarrow \mathbb{R}$ be Morse functions, and ρ_L, $\rho_{L'}$ Riemannian metrics on L and L'. Assume that f' has a single maximum, denoted by x'_n. Let $h : M \longrightarrow \mathbb{R}$ be a Morse function and ρ_M a Riemannian metric on M. Finally,

let $J \in \mathcal{J}$ be an almost complex structure. Assume that all these structures are generic so that the constructions in §2 work. Put $\mathcal{F} = (f, \rho_L)$, $\mathcal{F}' = (f', \rho_{L'})$, $\mathcal{H} = (h, \rho_M)$.

Given $x \in \mathrm{Crit}(f)$, $y' \in \mathrm{Crit}(f')$ and $k \in \mathbb{Z}$ consider the space of all tuples $(\mathbf{u}, v, R, \mathbf{u}')$ such that (see Figure 7):

(1) There exists $z \in L$, $A \in H_2^D(M, L)$, such that $\mathbf{u} \in \mathcal{P}_{\mathrm{prl}}(x, z; A; \mathcal{F}, J)$.
(2) There exists $z' \in L'$, $A' \in H_2^D(M, L')$, such that $\mathbf{u}' \in \mathcal{P}_{\mathrm{prl}}(z', y'; A'; \mathcal{F}', J)$.
(3) $1 < R < \infty$.
(4) $v : S^1 \times [1, R] \to M$ is a J-holomorphic map which satisfies $v(S^1 \times 1) \subset L'$, $v(S^1 \times R) \subset L$ and $v(-1, R) = z$, $v(1, 1) = z'$. Here we view S^1 as the unit circle in $\mathbb{C}$.
(5) The loop $v(S^1 \times 1)$ is contractible in M.
(6) $\mu(A) + \mu(A') + \mu([v]) = k$. Here the Maslov index $\mu([v])$ of the cylinder v is defined in an obvious way by trivializing $v^*T(M)$ over the cylinder $S^1 \times [1, R]$ and computing the difference of Maslov indices of the respective Lagrangian loops along the boundaries $S^1 \times 1$ and $S^1 \times R$.

We denote the space of such tuples $(\mathbf{u}, v, R, \mathbf{u}')$ by $\mathcal{P}_{\text{prl-cyl}}(x, y'; k; \mathcal{F}, \mathcal{F}', J)$.

For every cylinder v participating in an element $(\mathbf{u}, v, R, \mathbf{u}')$ as above we will now associate an element $\tau(v) \in \Lambda_{L,L'}$ as follows. Pick $1 < r_0 < R$ and choose a disk Q (in M) spanning the loop $v(S^1 \times r_0)$ (recall that the this loop is assumed to be contractible in M). By "dissecting" the cylinder v along the loop $v(S^1 \times r_0)$ we obtain two tubes, $T = v|_{S^1 \times [1, r_0]}$ and $T' = v_{S^1 \times [r_0, R]}$, one with a boundary component on L and the other with a boundary component on L'. By gluing Q to T and $\overline{Q}$ to T' ($\overline{Q}$ is Q with reversed orientation) we now obtain two disks w and w' with boundaries on L and L' respectively. ($\overline{Q}$ stands for Q with reversed orientation.) Obviously we have $\mu([w]) + \mu([w']) = \mu([v])$. We define

$$\tau(v) = t_0^{\mu([w])/N_L} t_1^{\mu([w'])/N_{L'}} \in \Lambda_{L,L'}.$$

It follows from the definition of the ring $\Lambda_{L,L'}$ (see also (27) that the element $\tau(v)$ does not depend on the choice of r_0 and the spanning disk Q.

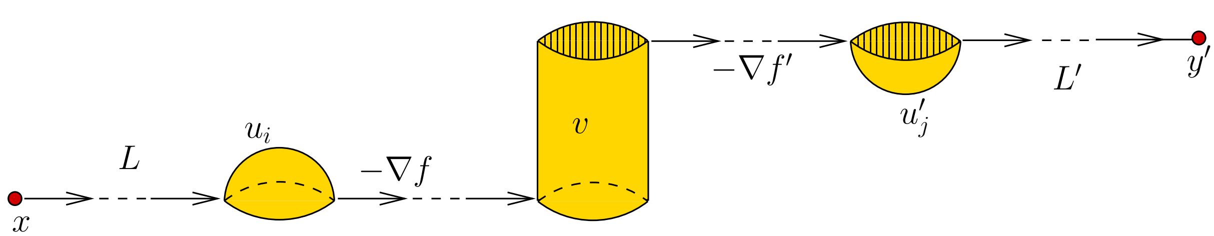

FIGURE 7. An element of the space $\mathcal{P}_{\text{prl-cyl}}(x, y'; k; \mathcal{F}, \mathcal{F}', J)$

We will need yet another moduli space which is defined as follows. Consider the space of all tuples $(\mathbf{u}, z, z', \mathbf{u}')$ such that (see Figure 8) :

(1) There exists $A \in H_2^D(M, L)$ such that $\mathbf{u} \in \mathcal{P}_{\text{prl}}(x, z; A; \mathcal{F}, J)$.
(2) There exists $A' \in H_2^D(M, L)$ such that $\mathbf{u}' \in \mathcal{P}_{\text{prl}}(z', y'; A'; \mathcal{F}', J)$.
(3) There exists $t > 0$ such that $\Phi_t(z) = z'$, where Φ_t is the negative gradient flow of the Morse function h with respect to ρ_M.
(4) $\mu(A) + \mu(A') = k$.

We denote the space of such tuples by $\mathcal{P}_{\text{prl-grad}}(x, y'; k; \mathcal{F}, \mathcal{F}', \mathcal{H}, J)$. The virtual dimension of both moduli spaces $\mathcal{P}_{\text{prl-cyl}}$ and $\mathcal{P}_{\text{prl-grad}}$ is:

$$\delta(x, y'; k) = |x| - |y'| - n + 1 + k.$$

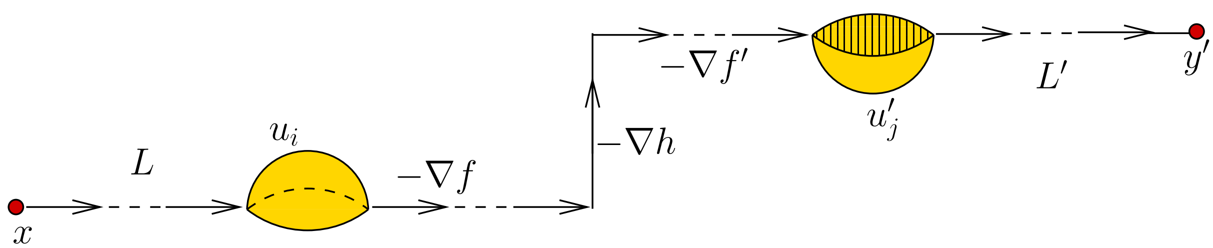

FIGURE 8. An element of the space $\mathcal{P}_{\text{prl-grad}}(x, y'; k; \mathcal{F}, \mathcal{F}', \mathcal{H}, J)$

Define a morphism $\Phi_{L,L'} : \mathcal{C}_*(\mathcal{F}, J; \Lambda_{L,L'}) \longrightarrow \mathcal{C}_{*-n+1}(\mathcal{F}', J; \Lambda_{L,L'})$ by:

$$\Phi_{L,L'}(x) = \sum_{k,y'} \Big(\sum_{(\mathbf{u},v,R,\mathbf{u}')} y' \, t_0^{\mu(\mathbf{u})/N_L} \tau(v) t_1^{\mu(\mathbf{u}')/N_{L'}} + \sum_{(\mathbf{u},z,z',\mathbf{u}')} y' \, t_0^{\mu(\mathbf{u})/N_L} t_1^{\mu(\mathbf{u}')/N_{L'}} \Big),$$

where the first sum is taken over all $k \in \mathbb{Z}$ and $y' \in \text{Crit}(f')$ with $\delta(x, y', k) = 0$; the second sum is taken over all $(\mathbf{u}, v, R, \mathbf{u}') \in \mathcal{P}_{\text{prl-cyl}}(x, y'; k; \mathcal{F}, \mathcal{F}', J)$; the third sum is taken over all $(\mathbf{u}, z, z', \mathbf{u}') \in \mathcal{P}_{\text{prl-grad}}(x, y'; k; \mathcal{F}, \mathcal{F}', \mathcal{H}, J)$.

Denote by $\widetilde{i}_L : \mathcal{C}_*(\mathcal{F}, J; \Lambda_{L,L'}) \longrightarrow C_*(\mathcal{H}; \Lambda_{L,L'})$ the quantum inclusion map (on the chain level) as defined by (12) in §2.6. The induced map in homology is i_L. Denote by $\widetilde{j}_{L'} : C_*(\mathcal{H}; \Lambda_{L,L'}) \longrightarrow \mathcal{C}_{*-n}(\mathcal{F}', J; \Lambda_{L,L'})$ the chain map defined by $\widetilde{j}_{L'}(a) = a \circledast x'_n$ (recall that x'_n is the single maximum of f'). Again, the induced map in homology is $j_{L'}$. For simplicity we denote the differentials of the complexes $\mathcal{C}(\mathcal{F}, J; \Lambda_{L,L'})$ and $\mathcal{C}(\mathcal{F}', J; \Lambda_{L,L'})$ by d and d' respectively.

Theorem 8.1.1 follows from the following.

Theorem* 8.2.1. *Suppose that $L \cap L' = \emptyset$. Then the following identity holds:*

$$\widetilde{j}_{L'} \circ \widetilde{i}_L = \Phi_{L,L'} \circ d + d' \circ \Phi_{L,L'}.$$

In other words, the chain map $\widetilde{j}_{L'} \circ \widetilde{i}_L$ is null homotopic. In particular the induced map in homology $j_{L'} \circ i_L$ vanishes.

Remark 8.2.2. A map similar to $\Phi_{L,L'}$ has been discussed before in the context of the cluster complex in [18] but it was used there to define a chain morphism (under certain assumptions) and not a chain homotopy. For some related earlier constructions see [24].

Note that for $\varphi \in \mathrm{Symp}_H(M, \omega)$ the map $j_{L'} \circ i_L$ vanishes iff $j_{\varphi(L')} \circ i_L$ vanishes. Therefore in proving Theorem 8.1.1 there is no loss of generality in assuming that $L \cap L' = \emptyset$ rather than $L \cap \varphi(L') = \emptyset$.

8.2.1. *Main ideas of the proof of Theorem 8.2.1.* In essence the proof follows the same standard scheme in Morse-Floer theory, as described in §3, i.e. compactifying certain 1-dimensional moduli spaces and deriving identities by counting the number of points in their boundaries.

Here is a more detailed account of the arguments. We need to introduce another type of moduli space. Denote by Φ_t^f, $\Phi_t^{f'}$ and Φ_t^h the negative gradient flows of (f, ρ_L), $(f', \rho_{L'})$ and (h, ρ_M) respectively. Let $x \in \mathrm{Crit}(f)$, $y' \in \mathrm{Crit}(f')$, and $k \in \mathbb{Z}$. Consider the space of all pairs $(\mathbf{u}, \mathbf{u}')$ where (see figure 9):

(1) $\mathbf{u} = (u_1, \ldots, u_l)$, $\mathbf{u}' = (u'_1, \ldots, u'_{l'})$ are two sequences of J-holomorphic disks $u_i : (D, \partial D) \longrightarrow (M, L)$, $u'_j : (D, \partial D) \longrightarrow (M, L')$. The disks $u_1, \ldots, u_{l-1}$ and $u'_2, \ldots, u'_{l'}$ are non-constant.
(2) $u_1(-1) \in W_x^u(f)$, $u'_{l'}(1) \in W_s^{y'}(f')$.
(3) For every $1 \leq i \leq l-1$ there exists $0 < t_i < \infty$ such that $\Phi_{t_i}^f(u_i(1)) = u_{i+1}(-1)$. For every $2 \leq j \leq l'$ there exists $0 < \tau_j < \infty$ such that $\Phi_{\tau_j}^{f'}(u'_{j-1}(1)) = u'_j(-1)$.
(4) There exists $0 < t < \infty$ such that $\Phi_t^h(u_l(0)) = u'_1(0)$.
(5) $\mu([\mathbf{u}]) + \mu([\mathbf{u}']) = k$.

We quotient the space of such elements by the obvious reparametrization groups. The resulting space is denoted by $\mathcal{P}_{\text{prl-prl}}(x, y'; k; \mathcal{F}, \mathcal{F}', \mathcal{H}, J)$. Its virtual dimension is $\delta(x, y'; k) = |x| - |y'| - n + 1 + k$.

Let $x \in \mathrm{Crit}(f)$, $y' \in \mathrm{Crit}(f')$, $k_0 \in N_L\mathbb{Z}$ and $k_1 \in N_{L'}\mathbb{Z}$ with $|x| - |y'| - n + k_0 + k_1 = 0$. In order to prove the chain homotopy formula in Theorem 8.2.1 we have to show that the coefficient of $y' t_0^{k_0/N_L} t_1^{k_1/N_{L'}}$ in $\widetilde{j}_{L'} \circ \widetilde{i}_L(x) - (\Phi_{L,L'} \circ d(x) + d' \circ \Phi_{L,L'}(x))$ vanishes. For this end, put $k = k_0 + k_1$ and consider the 1-dimensional moduli spaces $\mathcal{P}_{\text{prl-cyl}}(x, y'; k; \mathcal{F}, \mathcal{F}, J)$, $\mathcal{P}_{\text{prl-grad}}(x, y'; k; \mathcal{F}, \mathcal{F}, \mathcal{H}, J)$ and $\mathcal{P}_{\text{prl-prl}}(x, y'; k; \mathcal{F}, \mathcal{F}, \mathcal{H}, J)$.

The compactifications of these moduli spaces goes along the same lines as in §3.2 with the following additional types of boundary points:

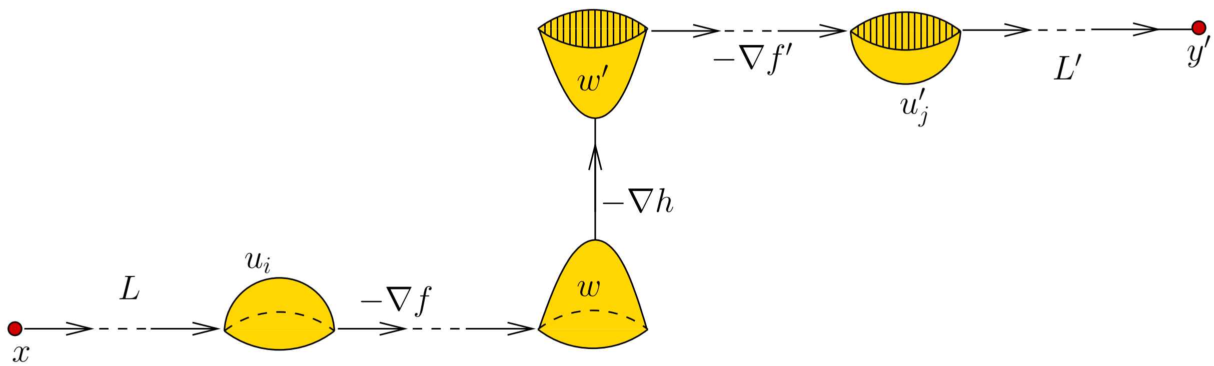

FIGURE 9. An element of the space $\mathcal{P}_{\text{prl-prl}}(x, y'; k; \mathcal{F}, \mathcal{F}', \mathcal{H}, J)$

a. The gradient trajectory of h involved in $\mathcal{P}_{\text{prl-prl}}(x, y'; k; \mathcal{F}, \mathcal{F}', \mathcal{H}, J)$ or in $\mathcal{P}_{\text{prl-grad}}(x, y'; k; \mathcal{F}, \mathcal{F}', \mathcal{H}, J)$ may break at a critical point of h.
b. A gradient trajectory of h involved in $\mathcal{P}_{\text{prl-prl}}(x, y'; k; \mathcal{F}, \mathcal{F}', \mathcal{H}, J)$ may shrink to a point. Note that this cannot happen for $\mathcal{P}_{\text{prl-grad}}(x, y'; k; \mathcal{F}, \mathcal{F}', \mathcal{H}, J)$ since L and L' are assumed to be disjoint.
c. The parameter R in elements $(\mathbf{u}, v, R, \mathbf{u}') \in \mathcal{P}_{\text{prl-cyl}}(x, y'; k; \mathcal{F}, \mathcal{F}', J)$ goes to ∞. The limit of the cylinder v in this case is two J-holomorphic disks w and w', one with boundary on L and one with boundary on L', attached to each other at an interior point. See figure 10. Note that the other type of degeneration $R \to 1$ is impossible here because $L \cap L' = \emptyset$.
d. Bubbling of a J-holomorphic disk coming from the cylinder v either with boundary on L or with boundary on L'. Note that bubbling of a J-holomorphic sphere from v may occur in general, but not in our case since we consider only 1-dimensional moduli spaces and such a bubbling would decrease the dimension to a negative one.

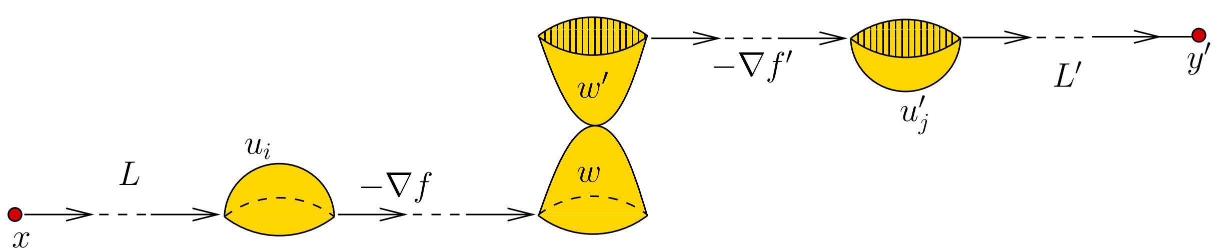

FIGURE 10. A J-holomorphic cylinder that converged to two disks

The above together with gluing arguments would then lead to a compactification of the 1-dimensional spaces $\mathcal{P}_{\text{prl-prl}}$, $\mathcal{P}_{\text{prl-grad}}$, $\mathcal{P}_{\text{prl-cyl}}$ into compact 1-dimensional manifolds with boundary. The identities needed to prove the homotopy formula in Theorem 8.2.1 would then follow by counting the number of points in the boundaries of these spaces. Note that since we want to show the vanishing of the coefficient of the monomial $y' t_0^{k_0/N_L} t_1^{k_1/N_{L'}}$ we actually have to restrict here only to those components of the spaces $\mathcal{P}_{\text{prl-cyl}}(x, y'; k; \mathcal{F}, \mathcal{F}, J)$, $\mathcal{P}_{\text{prl-grad}}(x, y'; k; \mathcal{F}, \mathcal{F}, \mathcal{H}, J)$, $\mathcal{P}_{\text{prl-prl}}(x, y'; k; \mathcal{F}, \mathcal{F}, \mathcal{H}, J)$ that contribute to this monomial. (In general, these spaces might contribute to other monomials of the type $y' t_0^{k_0'} t_1^{k_1'}$ with $k_0' + k_1' = k$, that are different than $y' t_0^{k_0} t_1^{k_1}$ in the ring $\Lambda_{L,L'}$.)

Another point which should be kept in mind within these arguments is that when $\delta_{\text{mod}}(a, x_n', y'; A') = 0$ and $\mu(A') > 0$, every element $(u_1, \ldots, u_l; r) \in \mathcal{P}_{\text{mod}}(a, x_n', y'; A; \mathcal{H}, \mathcal{F}', J)$ (see §2.5) must have $r = 1$, i.e. the disk with three marked points must be the first one. This follows from a straightforward transversality argument.

On the technical side, most of the steps of the proof indicated above can be carried out by essentially standard analytic techniques (versions of well known compactness theorems and gluing procedures). The only issue which remains to be rigorously clarified is how to achieve transversality for the spaces $\mathcal{P}_{\text{prl-cyl}}$, in particular for holomorphic cylinders. The difficulty occurs in the presence of holomorphic cylinders that are not somewhere injective.

Another approach which overcomes the transversality difficulties is to perturb the Cauchy-Riemann equation for the cylinders via Hamiltonian perturbations in the spirit of [1] (see also Chapter 8 of [29]). An even more natural type of perturbation is to replace the Morse function $h : M \to \mathbb{R}$ by a generic Hamiltonian function $H : M \times S^1 \to \mathbb{R}$. One can also replace the almost complex structure J by a time dependent one (though this is not really necessary here). The module action of $QH(M)$ on $QH(L')$ would now be replaced by the equivalent action of $HF(H)$ on $QH(L') \cong HF(L', L')$ (Here $HF(H)$ stands for the periodic-orbit Floer homology of H). The cylinder v would now satisfy a Floer-type equation (involving the Hamiltonian vector field of H), with Lagrangian boundary conditions etc. With these replacements the scheme of the proof presented above goes through with minor modifications. This approach will be further explored and developed in [11].

8.3. **Further generalizations.** When $L \cap L' \neq \emptyset$ the homotopy formula in Theorem 8.2.1 does not hold in general. The reason is that there are more types of boundary points for the spaces $\mathcal{P}_{prl-cyl}$ etc. than described in the preceding subsection. For example, when $L \cap L' \neq \emptyset$ it is possible to have a sequence $(\mathbf{u}_\nu, v_\nu, R_\nu, \mathbf{u}'_\nu) \in \mathcal{P}_{\text{prl-cyl}}(x, y'; k; \mathcal{F}, \mathcal{F}', J)$ with $R_\nu \longrightarrow 1$. A compactness argument shows that (generically) the limit of the cylinders

v_ν would look like a cylinder in which an arc connecting its two boundary components degenerated to a point p which lies in $L \cap L'$. Analytically, this limit object can also be viewed as a J-holomorphic strip with one boundary on L and one on L' connecting the point p with itself, i.e. a Floer connecting trajectory going from p to p.

By analyzing all the other possible boundary points for the relevant moduli spaces we obtain a correction term in the homotopy formula which takes into account information related to $L \cap L'$. More precisely, put $\mathcal{R} = \Lambda_{L,L'}$. Then we have:

$$\widetilde{j}_{L'} \circ \widetilde{i}_L - \widetilde{\chi}_{L,L'} = \Phi_{L,L'} \circ d + d' \circ \Phi_{L,L'},$$

where $\widetilde{\chi}_{L,L'} : \mathcal{C}_*(\mathcal{F}; \mathcal{R}) \longrightarrow \mathcal{C}_{*-n}(\mathcal{F}'; \mathcal{R})$ is a chain map which is the composition of two chain morphisms:

$$\widetilde{\chi}_{L,L'} : \mathcal{C}(\mathcal{F}; \mathcal{R}) \to CF(L, L'; \mathcal{R})^* \otimes_{\mathcal{R}} CF(L, L'; \mathcal{R}) \otimes_{\mathcal{R}} \mathcal{C}(\mathcal{F}'; \mathcal{R}) \xrightarrow{\langle -,- \rangle \otimes id} \mathcal{C}(\mathcal{F}'; \mathcal{R})$$

where $\langle -, - \rangle : A^* \otimes_{\mathcal{R}} A \to \mathcal{R}$ is the usual pairing of $A^* = Hom_{\mathcal{R}}(A, \mathcal{R})$ and A. (Note that we have to assume here that $N_L, N_{L'} \geq 3$ and that $\pi_1(L), \pi_1(L')$ both have torsion images in $\pi_1(M)$ in order for $HF(L, L')$ to be well defined and invariant.)

An immediate corollary of this is that if $j_{L'} \circ i_L \neq 0$ then $HF(L, L') \neq 0$. Similarly we can strengthen Corollaries 8.1.2 and 8.1.3 to conclude that not only $L \cap L' \neq \emptyset$ but also that $HF(L, L') \neq 0$. This direction will be further pursued in [11].

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